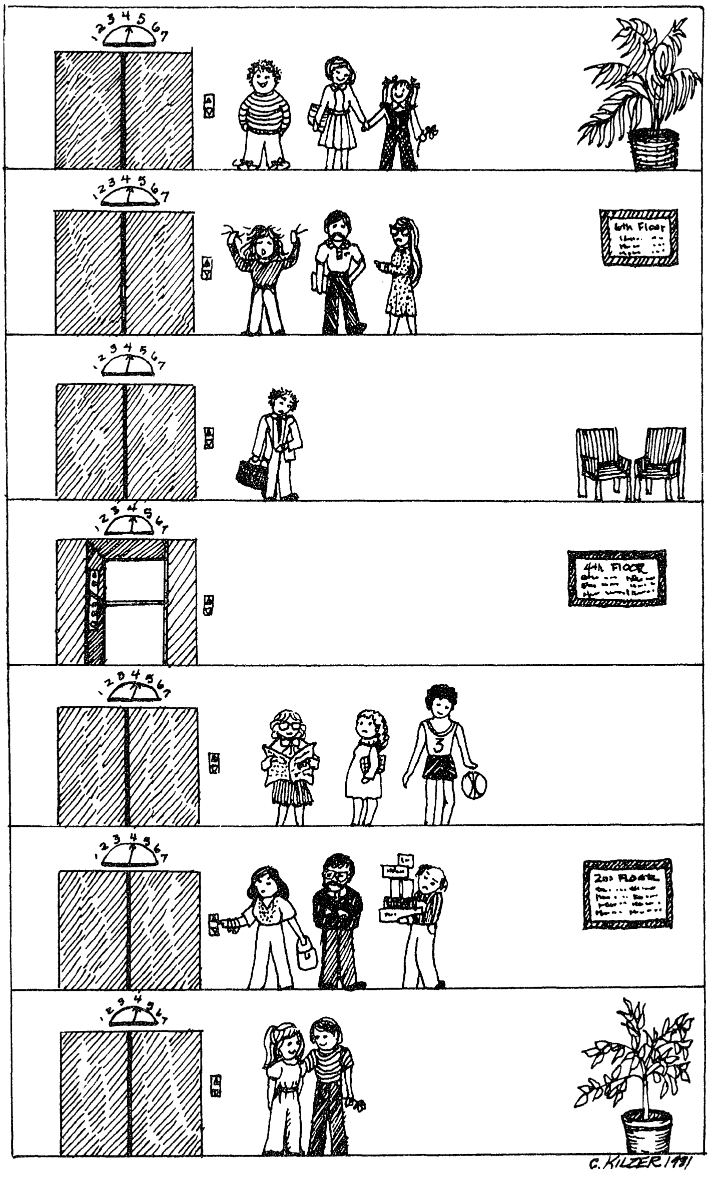
MATHEMATICS MAGAZINE



Vol. 55 No. 1
January, 1982

ELEVATOR WAITING • CHANGE OF VARIABLES IN $\int\int$ . . .
CONFIDENCE INTERVALS • GREEKS AND COMPUTERS

MATHEMATICS MAGAZINE

Volume 55, Number 1, January 1982
(ISSN—0025-570X)

COVER: Waiting... and waiting for an elevator. See p. 30. Design by C. Kilzer.

ARTICLES

NOTES

PROBLEMS

REVIEWS

NEWS AND LETTERS

EDITORIAL POLICY

Mathematics Magazine is a journal of collegiate mathematics which aims to provide inviting, informal mathematical exposition of interest to undergraduate students. Manuscripts accepted for publication in the *Magazine* should be written in a clear and lively expository style and stocked with appropriate examples and graphics. Our advice to authors is: say something new in an appealing way or say something old in a refreshing way. The *Magazine* is not a research journal and so the style, quality, and level of articles submitted for publication should realistically permit their use to supplement undergraduate courses. The editor invites manuscripts that provide insight into the history and application of mathematics, that point out interrelationships between several branches of mathematics and that illustrate the fun of doing mathematics.

New manuscripts should be sent to: Doris Schattschneider, Editor, Mathematics Magazine, Moravian College, Bethlehem, Pennsylvania 18018. Manuscripts should be prepared in a style consistent with the format of *Mathematics Magazine*. They should be typewritten and double spaced on $8\frac{1}{2}$ by 11 paper. Authors should submit the original and one copy and keep one copy as protection against possible loss. Illustrations should be carefully prepared on separate sheets of paper in black ink, the original without lettering and two copies with lettering added.

Authors planning to submit manuscripts should read the full statement of editorial policy which appears in the News and Letters section of this *Magazine*, Vol. 54, pp. 44–45. Additional copies of the policy are available from the Editor.

AUTHORS

Victor J. Katz ("Change of Variables in Multiple Integrals: Euler to Cartan") is Professor of Mathematics at the University of the District of Columbia. In 1978–79, he was fortunate to spend a sabbatical year at the Institute for the History and Philosophy of Science of the University of Toronto and the current article developed out of his research on the development of the theory of differential forms. Related papers have appeared in the May, 1981 issue of *Historia Mathematica* and this *Magazine*, Vol. 52.

Janet Bellcourt Pomeranz ("Confidence in Confidence Intervals") joined the faculty of the SUNY, Maritime College in 1971, where she is Professor of Mathematics. Her publications include papers in number theory, commutative rings, probability, and statistics. The idea for the present article originated when the author, as a student, witnessed an altercation between her instructor and a student over a particularly unenlightening definition of confidence interval.

ILLUSTRATIONS

Victor J. Katz is the collector of stamps which illustrate his article.
Vic Norton captured the thinkers on p. 24.
John Weaver provided the computer-drawn Figure 2 on p. 29.
C. Kilzer illustrates the art of elevator-waiting, pp. 31–36, and cover.
David Logothetti's agile acrobat performs in "$\forall$ and $\exists$," p. 41.
All other illustrations were provided by the authors.

BUSINESS INFORMATION. Mathematics Magazine is published by the Mathematical Association of America at Washington, D.C., five times a year in January, March, May, September, and November. The annual subscription price for Mathematics Magazine to an individual member of the Association is $8, included as part of the annual dues of $40. Students receive a 50% discount. Bulk subscriptions (5 or more copies to a single address) are available to colleges and universities for distribution to undergraduate students at a 35% discount. The library subscription price is $25.

Subscription correspondence and notice of change of address should be sent to A. B. Willcox, Executive Director, Mathematical Association of America, 1529 Eighteenth Street, N. W., Washington, D.C. 20036. Back issues may be purchased, when in print, from P. and H. Bliss Co., Middletown, Connecticut 06457.

Advertising correspondence should be addressed to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo, Buffalo, New York 14214.

Second class postage paid at Washington, D.C., and additional mailing offices.

ARTICLES

Change of Variables in Multiple Integrals: Euler to Cartan

From formalism to analysis and back; methods of proof come full circle.

Victor J. Katz
University of the District of Columbia
Washington, DC 20008

Leonhard Euler first developed the notion of a double integral in 1769 [7]. As part of his discussion of the meaning of a double integral and his calculations of such an integral, he posed the obvious question: what happens to a double integral if we change variables? In other words, what happens to $\int\int_{\mathbf{A}} f(x,y)\,dx\,dy$ if we let $x=x(t,v)$ and $y=y(t,v)$ and attempt to integrate with respect to t and v? The answer is provided by the change-of-variable theorem, which states that

$$\int\int_{\mathbf{A}} f(x,y)\,dx\,dy=\int\int_{\mathbf{B}} f(x(t,v),y(t,v))\left|\frac{\partial x}{\partial t}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial t}\right|dt\,dv \tag{1}$$

where the regions **A** and **B** are related by the given functional relationship between (x,y) and (t,v). This result, and its generalization to n variables, are extremely important in allowing one to transform complicated integrals expressed in one set of coordinates to much simpler ones expressed in a different set of coordinates. Every modern text in advanced calculus contains a discussion and proof of the theorem. (For example, see [5], [1], [18].)

Euler interpreted this result formally; namely, he considered $dx\,dy$ as an "area element" of the plane. So his aim was to show that his area element transformed into a new "area element" $\left|\frac{\partial x}{\partial t}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial t}\frac{\partial y}{\partial v}\right|dt\,dv$ under the given change of variables. Obviously, if we merely change coordinates by a translation, rotation, and/or reflection, the area element is transformed into a congruent one. So Euler noted that if t and v are new orthogonal coordinates related to x and y by

Leonhard Euler 1707–1783

a translation through constants a and b, a clockwise rotation through the angle θ whose cosine is m, and a reflection through the x-axis, i.e.,

$$x=a+mt+v\sqrt{1-m^2}$$

$$y=b+t\sqrt{1-m^2}-mv,$$

then $dx\,dy$ should be equal to $dt\,dv$. Unfortunately, when he performed the obvious formal calculation

$$dx=m\,dt+dv\sqrt{1-m^2},$$

$$dy=dt\sqrt{1-m^2}-m\,dv$$

and multiplied the two equations, he arrived at

$$dx\,dy=m\sqrt{1-m^2}\,dt^2+(1-2m^2)\,dt\,dv-m\sqrt{1-m^2}\,dv^2,$$

which, he noted, was obviously wrong and even meaningless (see FIGURE 1). Even more so, then, would a similar calculation be wrong if t and v were related to x and y by more complicated transformations. It was thus necessary for Euler to develop a workable method; i.e., one that in the above situation gives $dx\,dy=dt\,dv$ and, in general, gives $dx\,dy=Z\,dt\,dv$, where Z is a function of t and v.

To see how he arrived at his method, we must first consider his definition and calculation of double integrals. After noting that $\int\int Z\,dx\,dy$ means an "indefinite" double integral, i.e., a function of x and y which when differentiated first with respect to x and with respect to y gives $Z\,dx\,dy$, Euler proceeded to calculate "definite" integrals over specified planar regions **A** in the way familiar to calculus students. Thus, he wrote the integral as $\int dx \int Z\,dy$ and holding x constant, he integrated with respect to y between the functions $y=f_1(x)$ and $y=f_2(x)$ which bounded the region **A**; finally he integrated with respect to x between its minimum and maximum values in **A**. He interpreted this integral in the obvious way as a volume. In particular, he integrated

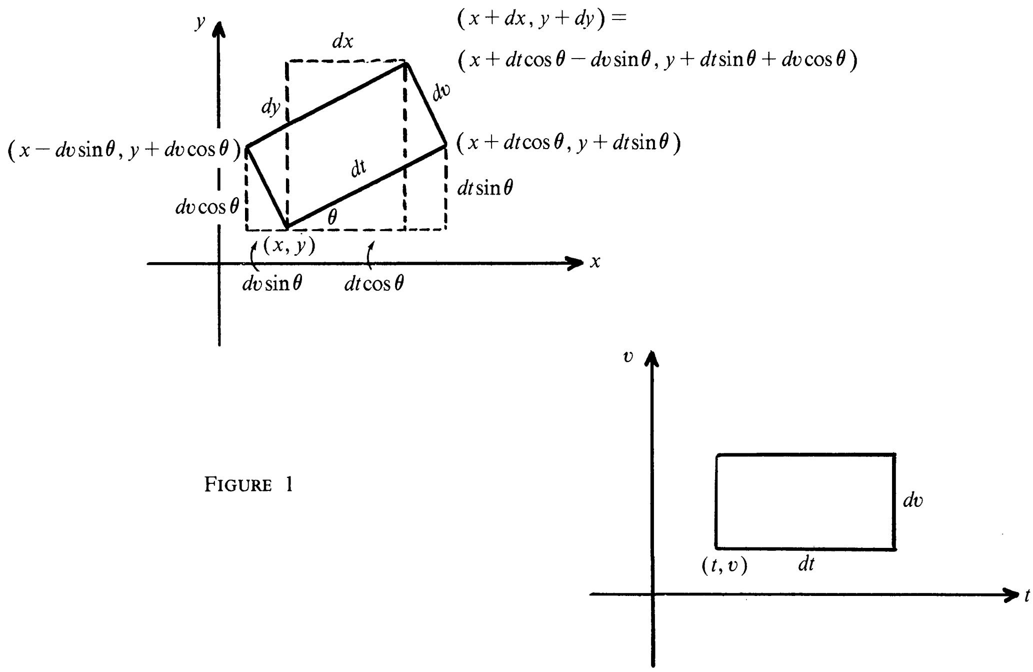


FIGURE 1

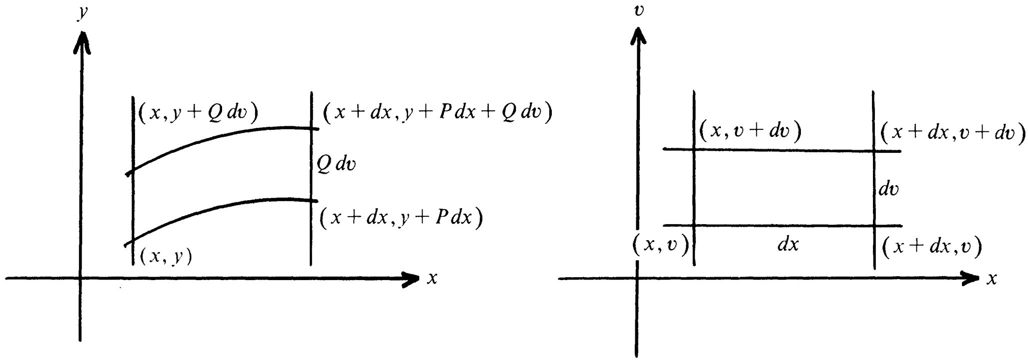


FIGURE 2

$\int\int\sqrt{c^2 - x^2 - y^2}\,dx\,dy$ over various regions to calculate volumes of portions of a sphere. Finally, he noted that $\int\int_{\mathbf{A}} dx\,dy$ is precisely the area of **A** and explicitly calculated the area of the circle given by $(x-a)^2 + (y-b)^2 = c^2$ to be πc^2.

Since the method of double integration involves leaving one variable fixed while dealing with the other, Euler proposed a similar method for the change-of-variable problem: change variables one at a time. First he introduced the new variable v and assumed that y could be represented as a function of x and v. So $dy = P\,dx + Q\,dv$ where P and Q are the appropriate partial derivatives. Now by assuming x fixed, he obtained $dy = Q\,dv$ and $\int\int dx\,dy = \int\int Q\,dx\,dv = \int dv \int Q\,dx$ (FIGURE 2). Next, he let x be a function of t and v and put $dx = R\,dt + S\,dv$. So by holding v constant, he calculated $\int dv \int Q\,dx = \int dv \int QR\,dt = \int\int QR\,dt\,dv$. This gave Euler the first solution to his problem: $dx\,dy = QR\,dt\,dv$.

Obviously, this was not completely satisfactory, since Q may well depend on x, and, in addition, the method was not symmetric. So Euler continued, now representing y as a function of t and v, hence $dy = T\,dt + V\,dv$. Then, formally, $dy = P\,dx + Q\,dv = P(R\,dt + S\,dv) + Q\,dv = PR\,dt + (PS + Q)\,dv$. So $PR = T$ and $PS + Q = V$, which gives $QR = VR - ST$. Euler's final answer was that $dx\,dy = (VR - ST)\,dt\,dv$. He noted again that simply multiplying the expressions for dx and dy together and rejecting the terms in dt^2 and dv^2 gives $(RV + ST)\,dt\,dv$, which differs by a sign from the correct answer. After a further note that one must always take the absolute value of the expression $VR - ST$ (since area is positive) he proceeded to confirm the correctness of his result through several increasingly complex examples.

This "proof" was typical of Euler's use of formal methods in many parts of his vast mathematical work. As a developer of algorithms to solve problems of various sorts, Euler has never been surpassed. (We can see that Euler's method, in modern notation, amounts to first factoring the transformation $x = x(t,v), y = y(t,v)$ into two transformations, the first being $x = x(t,v), v = v$ and the second $x = x, y = y(x,v)$. This can be done by "solving" $x = x(t,v)$ for t in the form $t = h(x,v)$ and then writing $y = y(h(x,v),v)$. Then $P = y_1\frac{\partial h}{\partial x}, Q = y_1\frac{\partial h}{\partial v} + y_2$, $R = x_1, S = x_2, T = y_1$ and $V = y_2$, where subscripts denote partial derivatives. Since $x(h(x,v),v) = x$ and $h(x(t,v),v) = t$, we calculate that $\frac{\partial h}{\partial x}x_1 = 1$ and $\frac{\partial h}{\partial x}x_2 + \frac{\partial h}{\partial v} = 0$, so $PR = T$ and $PS + Q = V$.)

In 1773 J. L. Lagrange also had need of a change-of-variable formula—this time for triple integrals [**12**]. He was interested in determining the attraction which an elliptical spheroid exercised on any point placed on its surface or in the interior. Since the general expression for attraction at any point was well known, the difficulty lay in integrating over the entire body. Even though the problem had already been solved geometrically, Lagrange, as part of his general philosophy of treating mathematics analytically, attempted a different solution.

To solve his problem, Lagrange had to calculate a triple integral. Since, following Euler's

Joseph-Louis Lagrange 1736–1813

method, this had to be done by first holding two variables constant, integrating with respect to the third from one surface of the body to another, then evaluating the ensuing double integrals, he was quickly led to very complicated integrands. He realized that new coordinates were needed to replace the rectangular ones in order to make the integration tractable. Thus he proceeded to develop a general formula for changing variables in a triple integral. Lagrange's method was similar to Euler's in that he let vary only one variable at a time, but the details differed.

Given, then, x, y, and z as functions of new variables p, q, r, Lagrange wrote

$$\begin{aligned} dx &= A\,dp + B\,dq + C\,dr \\ dy &= D\,dp + E\,dq + F\,dr \\ dz &= G\,dp + H\,dq + I\,dr \end{aligned} \tag{2}$$

where $A, B, \ldots, I$ are, of course, the appropriate partial derivatives. His aim was to calculate the volume of the infinitesimal parallelepiped $dx\,dy\,dz$ (the "volume element") in terms of $dp\,dq\,dr$. To do this, he calculated each "difference" (i.e., edge of the parallelepiped) separately, regarding the other two variables as constant. First x and y are held constant; thus $dx = 0$ and $dy = 0$; the first two equations in (2) become

$$\begin{aligned} A\,dp + B\,dq + C\,dr &= 0 \\ D\,dp + E\,dq + F\,dr &= 0. \end{aligned}$$

Lagrange solved these two equations for dp and dq in terms of dr and substituted in the expression for dz in (2) to get

$$dz = \frac{G(BF-CE)+H(CD-AF)+I(AE-BD)}{AE-BD}\,dr.$$

Next, x and z are assumed constant and only y varies; so $dx = 0$ and $dz = 0$. It follows immediately that $dr = 0$ and $A\,dp + B\,dq = 0$; therefore, $dp = -(B/A)\,dq$ and

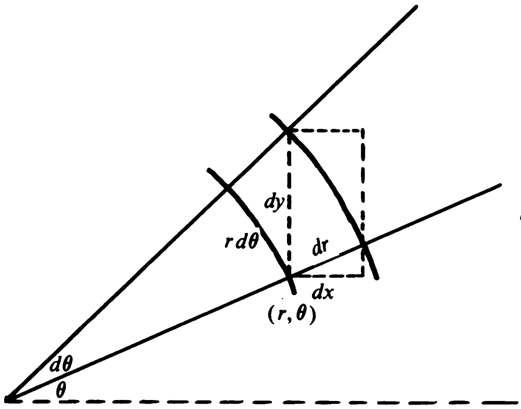
$$dy = \frac{AE-BD}{A}\,dq.$$

Finally, y and z are taken as constant, so $dy = 0$ and $dz = 0$. Thus $dr = 0$ and $dq = 0$, which implies that $dx = A\,dp$. By multiplying together the expressions obtained for dx, dy, and dz, Lagrange calculated his result:

$$dx\,dy\,dz = (AEI + BFG + CDH - AFH - BDI - CEG)\,dp\,dq\,dr. \tag{3}$$

This is, of course, our standard formula. The result for three-dimensional integrals is analogous to (1), and in modern notation, is written as

$$\iiint_{\mathbf{A}} f(x,y,z)\,dx\,dy\,dz = \iiint_{\mathbf{B}} f(x(p,q,r),y(p,q,r),z(p,q,r))\left|\frac{\partial(x,y,z)}{\partial(p,q,r)}\right| dp\,dq\,dr$$



$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dx = \cos\theta\, dr - r\sin\theta\, d\theta$$
$$dy = \sin\theta\, dr + r\cos\theta\, d\theta$$
$$dx = 0 \Rightarrow dy = r\sec\theta\, d\theta$$
$$dy = 0 \Rightarrow dx = \cos\theta\, dr$$
$$dx\, dy = r\, dr\, d\theta$$

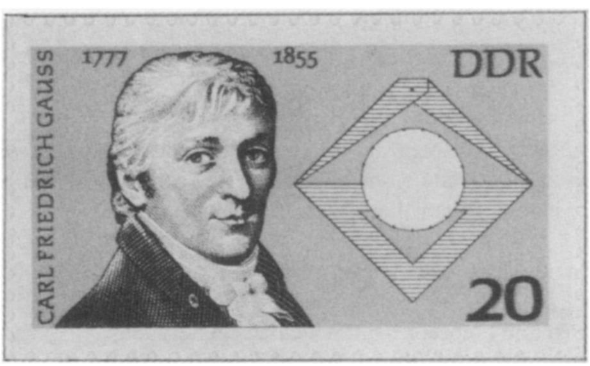
FIGURE 3

where $\dfrac{\partial(x,y,z)}{\partial(p,q,r)}$ is the functional determinant of x, y, z with respect to p, q, r. (FIGURE 3 illustrates Lagrange's idea for the case of two variables and polar coordinates.)

We note that Lagrange, like Euler, dealt with the differential forms formally; there is absolutely no infinitesimal approximation that we would require in a similar proof today. But this formalism is typical of some of Lagrange's other work, in particular, his attempt to develop the calculus without limits by the use of algebra and infinite series [11], [13]. Also like Euler, Lagrange noted that the most obvious thing to do to try to obtain the change-of-variable formula would be to multiply together the original expressions (2) for *dx*, *dy*, and *dz*. However, he wrote, this product would contain squares and cubes of *dp*, *dq*, and *dr* and so would not be valid in an expression of a triple integral. Hence he had to use the step-by-step formal approach already outlined.

Lagrange applied his result to the case of spherical coordinates and was then able to perform the integrations he needed. Similarly, A. Legendre [15] and Pierre S. Laplace [14] soon after used essentially the same method to get similar results. These men were also interested in the change-of-variable formula in order to determine the attraction exercised by solids of various shapes, for which they needed to compute complicated integrals.

In 1813 Carl F. Gauss gave a geometric argument for a special case of the change-of-variable theorem for two variables, although in a somewhat different context [8]. Gauss' method of proof contrasts sharply with that of Euler. Gauss was developing the idea of a surface integral in connection with studying attractions. As part of this he gave a method for finding the element of surface in three-space so that he could integrate over such a surface. He started by parametrizing



Carl Friedrich Gauss 1777–1855

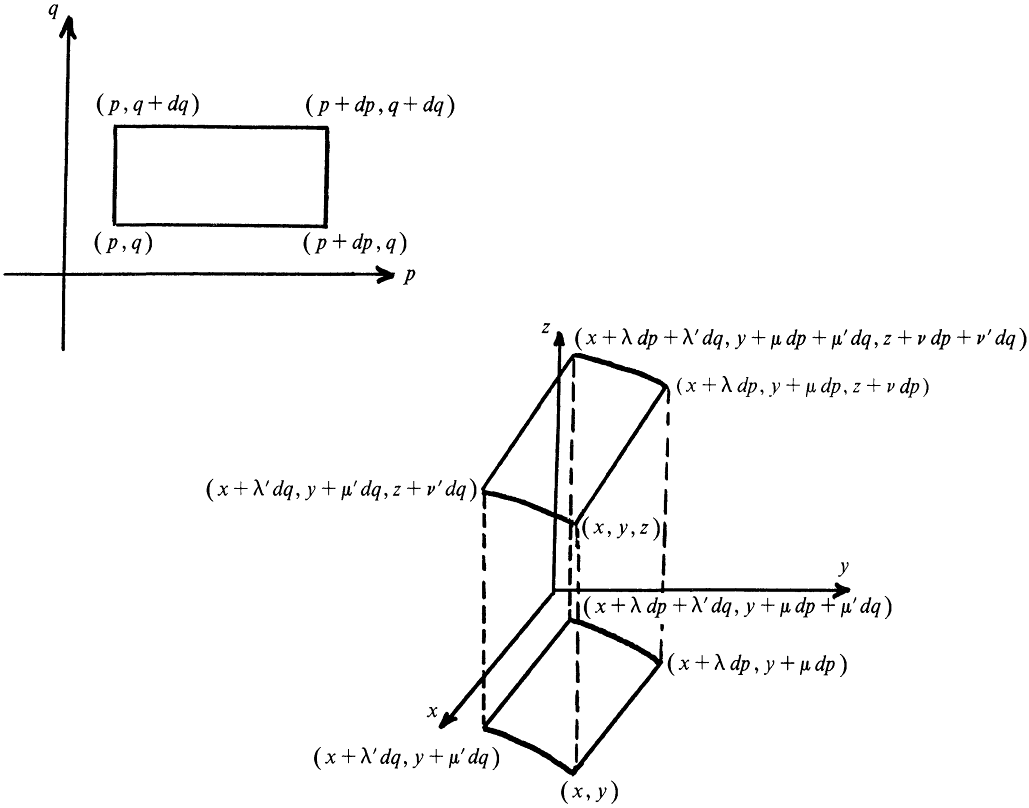


FIGURE 4

the surface using three functions x, y, z of the two variables p, q. He then noted that given an infinitesimal rectangle in the p-q plane whose vertices were (p, q), $(p + dp, q)$, $(p, q + dq)$, $(p + dp, q + dq)$, there was a corresponding "parallelogram" element in the surface whose vertices were (x, y, z), $(x + \lambda\, dp, y + \mu\, dp, z + \nu\, dp)$, $(x + \lambda'\, dq, y + \mu'\, dq, z + \nu'\, dq)$, and $(x + \lambda\, dp + \lambda'\, dq, y + \mu\, dp + \mu'\, dq, z + \nu\, dp + \nu'\, dq)$, where

$$\begin{aligned} dx &= \lambda\, dp + \lambda'\, dq \\ dy &= \mu\, dp + \mu'\, dq \\ dz &= \nu\, dp + \nu'\, dq. \end{aligned} \tag{4}$$

(One can easily calculate the above result from the definitions and properties of the relevant partial derivatives.) It follows that the projection of the infinitesimal parallelogram onto the x-y plane is the parallelogram whose vertices are (x, y), $(x + \lambda\, dp, y + \mu\, dp)$, $(x + \lambda'\, dq, y + \mu'\, dq)$, $(x + \lambda\, dp + \lambda'\, dq, y + \mu\, dp + \mu'\, dq)$ and whose area is clearly $\pm(\lambda\mu' - \mu\lambda')\, dp\, dq$. (See FIGURE 4.) Gauss was therefore able to compute the element of surface area as $dp\, dq((\mu\nu' - \nu\mu')^2(\nu\lambda' - \lambda\nu')^2(\lambda\mu' - \mu\lambda')^2)^{1/2}$ and thus to integrate this over the p-q region corresponding to his surface. (In this paper, Gauss used his special cases of the divergence theorem and his parametric method for calculating a surface element to evaluate certain "surface integrals" for the case of an ellipsoid given by $x = A\cos(p), y = B\sin(p)\cos(q), z = C\sin(p)\sin(q)$ for $0 \leqslant p \leqslant \pi$, $0 \leqslant q \leqslant 2\pi$.)

If we let $z = 0$ so that the "surface" is part of the x-y plane, then Gauss' argument shows that the new "area element" is $|\lambda\mu' - \mu\lambda'|\, dp\, dq$, hence that $\int\int dx\, dy = \int\int |\lambda\mu' - \mu\lambda'|\, dp\, dq$, a special case of the change-of-variable theorem from which the general case may easily be derived. Gauss' argument differs considerably from those of Euler and Lagrange. He essentially made use of analytic and geometric methods instead of using the formal approach of his predecessors. But as was typical of Gauss, he did not provide all the steps necessary to complete his analytic argument, especially since he was dealing with infinitesimals. The missing parts can, however, be readily supplied.

The next mathematician to break new ground in this field was Mikhail Ostrogradskii, in 1836. A Russian mathematician who studied in France in the 1820's, he later returned to St. Petersburg where he produced many works in applied mathematics. Unfortunately, some of his most important discoveries appear to have been totally ignored, at least in Western Europe. Not only did he give the first generalization of the change-of-variable theorem to n variables, but he also first proved and later generalized the divergence theorem **[10]**, wrote integrals of n-forms over n-dimensional "hypersurfaces," and, as we shall see below, gave the first proof of the change-of-variable theorem for double integrals using infinitesimal concepts. All of these results were eventually repeated by other mathematicians with no credit to Ostrogradskii.

In his 1836 paper **[16]**, Ostrogradskii generalized to n dimensions the change-of-variable theorem and Lagrange's proof of it. Given that $X, Y, Z,\ldots$ are all functions of $x, y, z\ldots$, Ostrogradskii first calculated $dX, dY, dZ,\ldots$ in terms of $dx, dy, dz,\ldots$. Then by holding all variables except X constant, he had $dY = dZ = \cdots = 0$, so he could solve for dX in terms of dx by using determinants; continuing with each variable in turn he calculated expressions for $dY, dZ,\ldots$ in terms of $dy, dz,\ldots$ and by multiplying showed that $dX\,dY\,dZ\ldots = \Delta\, dx\, dy\, dz\ldots$ where Δ is the functional determinant of $X, Y, Z,\ldots$ with respect to $x, y, z,\ldots$. Ostrogradskii did not state this result as a formula for transforming multiple integrals, but he did apply it to convert a hypersurface integral with $n+1$ terms of the form $dx\, dy\ldots$, to an ordinary n-dimensional integral in n new variables.

Both Carl Jacobi **[9]** and Eugene Catalan **[4]** published papers in 1841 giving clearly the general change-of-variable theorem for n-dimensional integrals. Catalan's proof was also similar to Lagrange's in its use of formal manipulations on one variable at a time. Jacobi's paper was the culmination of a series of articles concerning this theorem; it contained additional results such as the multiplication rule for the composition of several changes of variable. Jacobi's work was referred to shortly thereafter by Cauchy and soon his name became tied to the theorem. In fact, the functional determinant Δ is now known as the Jacobian rather than the "Ostrogradskian."

Two years after his 1836 paper, Ostrogradskii published in **[17]** a proof of the change-of-variable formula in two variables which used the same basic idea as had Gauss. He first criticized the proofs of Euler and Lagrange, and, by implication, his own earlier proof. He claimed that, assuming that x and y were functions of u and v, if one first used $dx = 0$ to solve for dy in terms of du (that is, to evaluate one side of the differential rectangle) one could not then assume that du would be 0 when one tried to evaluate dx by setting $dy = 0$ (to find the other side of the rectangle). In fact, he wrote, you would have to use a new set of differentials, δu and δv, in evaluating the other side, and, once you did that, you came up with an incorrect result.

Mikhail Ostrogradskii 1801–1861

So Ostrogradskii returned to the meaning of $\int\int V dx\, dy$ as a sum of differential elements. Using a method similar to that of Gauss, although staying strictly in two dimensions, he proceeded to recalculate the area of these elements. He carefully chose each element to be bounded by two curves where u was constant and two curves where v was constant. If ω denotes the area of such an element, he noted that by the definition of the definite integral, $\int\int V dx\, dy = \int\int V\omega$. It is easy to calculate ω (see FIGURE 5) since the four vertices have coordinates (x, y), $\left(x + \frac{\partial x}{\partial u}du, y + \frac{\partial y}{\partial u}du\right)$, $\left(x + \frac{\partial x}{\partial v}dv, y + \frac{\partial y}{\partial v}dv\right)$, and $\left(x + \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv, y + \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$. By elementary geometry, the area of this parallelogram is $\pm\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) du\, dv$ and so the integral formula becomes

$$\int\int V dx\, dy = \pm\int\int V\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right) du\, dv.$$

Ostrogradskii further noted that this method could be easily extended to three dimensions but not more, since there is not a corresponding geometrical result in four dimensions. We must note, of course, that Ostrogradskii had not explicitly justified using the standard formula for the area of a parallelogram when, in fact, the area is actually that of a "curvilinear" parallelogram. However, it was common practice in that time (as we noted also about Gauss' proof), to ignore explicit arguments about infinitesimal approximation.

Only four years later, a proof similar to that of Ostrogradskii appeared in Augustus DeMorgan's text *Differential and Integral Calculus* [**6**], one of the first "analytic" textbooks to appear in English. It is doubtful that DeMorgan had read Ostrogradskii's work, for his approach is somewhat different; he was considering how to calculate a double integral over a plane region bounded by four curves, where the standard method of integrating, first with respect to one variable between two functions of the other and then with respect to the second between constant limits, will not work. But his method of attack, via the definition of the double integral as a limit, the division of the given region into subregions bounded by curves where u was constant and where v was constant, and the calculation of areas of curvilinear quadrilaterals, is very close to that of Ostrogradskii. DeMorgan went even further, however, to provide detailed reasoning as to why the errors of approximation—third order infinitesimals—may be safely ignored.

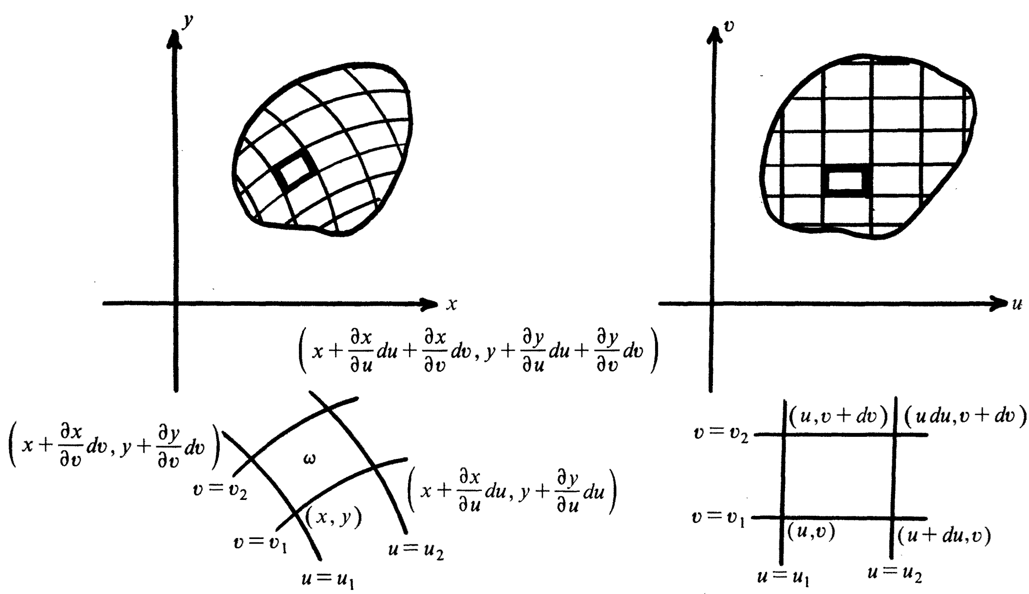


FIGURE 5

It is also interesting that DeMorgan prefaced his results by stating that Legendre's proof (which was identical to that of Lagrange) was "so obscure in its logic as to be nearly unintelligible, if not dubious."

Ostrogradskii and DeMorgan, then, had moved away from the formal symbolic approach of Euler and Lagrange. But we should emphasize that the former had not justified equating the "elements of area" $dx\,dy$ and $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$ themselves, as the latter had attempted to do. They had only showed the equality of the integrals over the appropriate regions. A new justification for the formal symbolic approach only came with Elie Cartan and his theory of differential forms.

Beginning in the mid 1890's, Cartan wrote a series of papers in which he formalized the subject of differential forms, namely the expressions which appear under the integral sign in line and surface integrals. As part of this formalization, he used the Grassmann rules of exterior algebra for calculations with such forms. In a paper of 1896 [**2**], as an example of such a calculation, he was able to do what Euler could not; namely, if $x=x(t,v)$ and $y=y(t,v)$, he could multiply $dx=\frac{\partial x}{\partial t}dt+\frac{\partial x}{\partial v}dv$ and $dy=\frac{\partial y}{\partial t}dt+\frac{\partial y}{\partial v}dv$ using the rules $dt\,dt=dv\,dv=0$ and $dt\,dv=-dv\,dt$ to show that

$$dx\,dy=\left(\frac{\partial x}{\partial t}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial t}\right)dt\,dv.$$

In 1899 [**3**], Cartan went into much more detail on the rules for operating with these differential forms. And again, one of his first examples was the change-of-variable formula.

As a final point, we note that proofs using the methods of Euler, Lagrange, and Ostrogradskii all appeared in textbooks through the first third of the twentieth century. There were, naturally, attempts to make all three methods more rigorous. A readily available example of this (for the proofs of Euler and Ostrogradskii) occurs in Courant's *Differential and Integral Calculus* [**5**]. Most current textbooks, on the other hand, use an entirely different proof based on Green's theorem.

References

[1] R. C. Buck, Advanced Calculus, 2nd ed., McGraw-Hill, New York, 1965, p. 296 ff.

[2] E. Cartan, Le principe de dualité et certaines intégrales multiples de l'espace tangentiel et de l'espace réglé, Bull. Soc. Math. France, 24 (1896) 140–177; Oeuvres (2) 1, 265–302.

[3] ______, Sur certaines expressions différentielles et sur le problème de Pfaff, Annales École Normale, 16 (1899) 239–332; Oeuvres (2) 1, 303–397.

[4] E. Catalan, Sur la transformation des variables dans les intégrales multiples, Mémoires couronnés par l'Académie Royale des Sciences et Belles-Lettres de Bruxelles, 14 (1841) 1–48.

[5] R. Courant, Differential and Integral Calculus, vol. 2, Interscience, New York, 1959, p. 247 ff.

[6] A. DeMorgan, The Differential and Integral Calculus, Robert Baldwin, London, 1842.

[7] L. Euler, De formulis integralibus duplicatis, Novi comm. acad. scient. Petropolitanae, 14 (1769) 72–103; Opera (1) 17, 289–315.

[8] C. Gauss, Theoria attractionis corporum sphaeroidicorum ellipticorum homogeneorum methodo novo tractata, Comm. soc. reg. scient. Gottingensis, 2 (1813): Werke 5, 1–22.

[9] C. Jacobi, De determinantibus functionalibus, Crelle, 22 (1841) 319–359; Werke 3, 393–438.

[10] V. Katz, The history of Stokes' theorem, this MAGAZINE, 52 (1979) 146–156.

[11] J. Lagrange, Théorie des Fonctions Analytiques, Paris, 1797; Oeuvres 9.

[12] ______, Sur l'attraction des sphéroïdes elliptiques, Nouveaux Mémoires de l'Académie Royale des Sciences et Belles-Lettres de Berlin, (1773); Oeuvres 3, 619–658.

[13] ______, Leçons sur le Calcul des Fonctions, Paris, 1801.

[14] P. Laplace, Traité de Mécanique Céleste, Paris, 1799; Oeuvres 2.

[15] A. Legendre, Mémoire sur les intégrales doubles, Histoire de l'Académie Royale des Sciences avec les Mémoires de Mathématique et de Physique, (1788) 454–486.

[16] M. Ostrogradskii, Mémoire sur le calcul des variations des intégrales multiples, Mémoires de l'Académie Impériale des Sciences de St. Petersbourg, (6) 3 (part 1) (1836), 36–58; also in Crelle, 15 (1836) 332–354.

[17] ______, Sur la transformation des variables dans les intégrales multiples, Mémoires de l'Académie Impériale des Sciences de St. Petersbourg, cl. math., (1838) 401–407.

[18] A. Taylor and W. R. Mann, Advanced Calculus, 2nd ed., Xerox, Lexington, 1972, p. 490 ff.

Confidence in Confidence Intervals

An exoteric view of confidence intervals that involves subjective probability and makes use of computer-simulated sampling.

Janet Bellcourt Pomeranz
State University of New York Maritime College
Fort Schuyler, Bronx, NY 10465

Picture this. The long rays of late afternoon sunlight are filtering through neatly leveled venetian blinds. Chalk and erasers are in readiness alongside horizontally erased blackboards. Every student in the classroom is flipping through the pages of a textbook on probability and statistics, some casually, a few with great intensity. A bell rings, the professor enters, class is about to begin. The lesson of the day is confidence intervals.

Squinting at yellowed notes and turning as if to address the blackboard, the professor writes on the board as he solemnly recites:

> If for a given α
> the relation $P(\theta_1 \leqslant \theta \leqslant \theta_2) = 1 - \alpha$ holds
> we say that the interval $[\theta_1, \theta_2]$ is a
> $100(1-\alpha)\%$ confidence interval for the parameter θ.

Before the professor has finished transcribing the statement from his notes to the board, the students have swung into action and are busy copying the definition into their spiral notebooks. One student, however, balks. Having attracted attention by clearing his throat and waving frantically, he sputters, "Excuse me, Professor, but what *is* a confidence interval?" The professor peers myopically in the direction of the student. Gesturing toward the blackboard, with great deliberation he repeats the definition verbatim.

Exasperated, the student pleads, "But, Professor, you know I cannot think *abstractly*." Again the professor tries to locate the refractory student, his eyes narrowing. All at once the room has become quiet. With patience, or is it weariness, he rejoins: "On the contrary, you cannot think *except* abstractly."

This story is true, as my own spiral notebook will attest. Why do I recount the anecdote here? Is it because the student's anguish impresses me as genuine, and I wish to commiserate somehow with students everywhere? Perhaps. The real reason, though, is to demonstrate the need for a less abstruse concept of confidence interval. The professor's definition is neat and unambiguous but is hardly enlightening to the average student.

In this paper I take a more tangible approach to confidence intervals, through hypothesis testing. The interpretation of confidence interval which I propose is linked to subjective probability rather than classical probability and is, I believe, simpler than the standard textbook interpretation. As a means of demonstrating in a concrete way the salient features of confidence intervals, I utilize a BASIC program in order to simulate sampling on a computer. Besides the program itself, I include in this paper the results of several demonstrations involving computer-simulated sampling.

From the black-or-white mentality of hypothesis testing to interval estimation

"Do you think that discipline at this college should be stricter?" When I ask this question of

FIGURE 1. **"Excuse me, Professor, but what *is* a confidence interval?"**

my statistics class the answer is a resounding, "No." "Do you suppose all students feel this way?" In response, one member of the class ventures the opinion that possibly one in ten students would favor stricter discipline. The stage has now been set for a lesson on hypothesis testing and on the related theme of confidence intervals.

To test hypotheses concerning the proportion p of students in the college population favoring stricter discipline, we take a simple random sample of size n and compute the sample proportion, $\hat{p} = x/n$ where x is the number of students in the sample favoring stricter discipline. Under certain rather general conditions, the sampling distribution of $\hat{p}$ is approximately normal with mean p and standard deviation $\sqrt{p(1-p)/n}$. When are conditions right? Loosely speaking, if n is large enough and p is not too close to either 0 or 1. More precisely, if $np > 5$ when $p \leqslant 0.5$, and $n(1-p) > 5$ when $p > 0.5$, the normal approximation to $\hat{p}$ is satisfactory [7, p. 114].

We're interested in testing the hypothesis H_0: $p = 0.10$ against the alternative hypothesis H_A: $p \neq 0.10$. The testing procedure calls for our assuming at the outset that H_0 is true, that is, that $p = 0.10$. By choosing $n = 100$, we have $np = 10$ and easily meet the requirement that $np > 5$ when $p \leqslant 0.5$. Accordingly, the test statistic

$$Z = (\hat{p} - 0.10)/\sqrt{(0.10)(0.90)/100}$$

is approximately standard normal. Having decided to test at the 5% level of significance, we agree to reject H_0 if and only if $|Z| > 1.960$.

Our poll reveals that in a simple random sample of size 100, exactly 15 students favor stricter discipline. A few of the more impetuous members of my class jump to the conclusion that the population proportion p is not 0.10. After all, if p were 0.10, would $\hat{p}$ be as large as 0.15? On calculating the test statistic

$$Z = (0.15 - 0.10)/\sqrt{(0.10)(0.90)/100} = 1.667,$$

though, we are forced to conclude that at the 5% significance level our findings *are* consistent with the hypothesis that $p = 0.10$. Still, most students are inclined to adjust upward the original estimate of 0.10 for p. Someone notices that the computed value of Z is a little larger than 1.645, the cutoff value for $|Z|$ at the 10% significance level, and I concede that at that level we do reject the hypothesis that $p = 0.10$. At this point some students openly express skepticism of the hypothesis testing process. Is the population proportion 0.10, or isn't it? They simply do not like a testing procedure which on the basis of the same sample leads either to one conclusion or to its opposite, depending on one's choice of significance level for the test.

Rather than engage in the black-or-white type of thinking that characterizes hypothesis testing, statisticians often turn to confidence intervals. Instead of deciding whether the value of a parameter is equal to some preassumed value, we calculate an interval which has a preassigned probability of containing the unknown parameter. For example, a 90% confidence interval for the population proportion p consists of all values within $1.645\sqrt{\hat{p}(1-\hat{p})/n}$ of the sample proportion $\hat{p}$. To find a 95% or 99% confidence interval we simply replace 1.645 by 1.960 and 2.576, respectively. Using the sample results of $\hat{p}=0.15$ and $n=100$, we compute 90%, 95%, and 99% confidence intervals for p as follows (see FIGURE 2):

$$0.15 \pm 1.645\sqrt{(0.15)(0.85)/100} = [0.09, 0.21],$$

$$0.15 \pm 1.960\sqrt{(0.15)(0.85)/100} = [0.08, 0.22],$$

$$0.15 \pm 2.576\sqrt{(0.15)(0.85)/100} = [0.06, 0.24].$$

There is a close connection between confidence intervals and tests of hypotheses [**11**, p. 278], [**12**, p. 116], [**15**, p. 202], [**20**, p. 114]. Suppose, for example, that T is a 95% confidence interval for μ, the mean of a normal random variable, with σ^2 known. Then for the same observed sample, we accept the hypothesis H_0: $\mu=\mu_0$ as opposed to the hypothesis H_A: $\mu\neq\mu_0$ at the 5% level of significance if and only if $\mu_0 \in T$. More generally, the acceptance region for a significance level α is exactly the same as the $100(1-\alpha)\%$ confidence interval for μ. Notice in the example given above that $0.10 \in [0.08, 0.22]$; that is, the hypothesized value for p lies in the 95% confidence interval. So we accept H_0: $p=0.10$ versus H_A: $p\neq 0.10$ at the 5% significance level. But the hypothesized value, 0.10, is also in the 90% confidence interval, and cursory analysis would lead us to accept H_0 at the 10% level too, contrary to our earlier conclusion. Why this discrepancy? The answer lies in the fact that in our example σ^2 is unknown because we don't know the value of p. While it is true that in the testing situation we assume under H_0 that the real value of p is 0.10, in the interval estimation situation the real value of p is not known. In computing a confidence

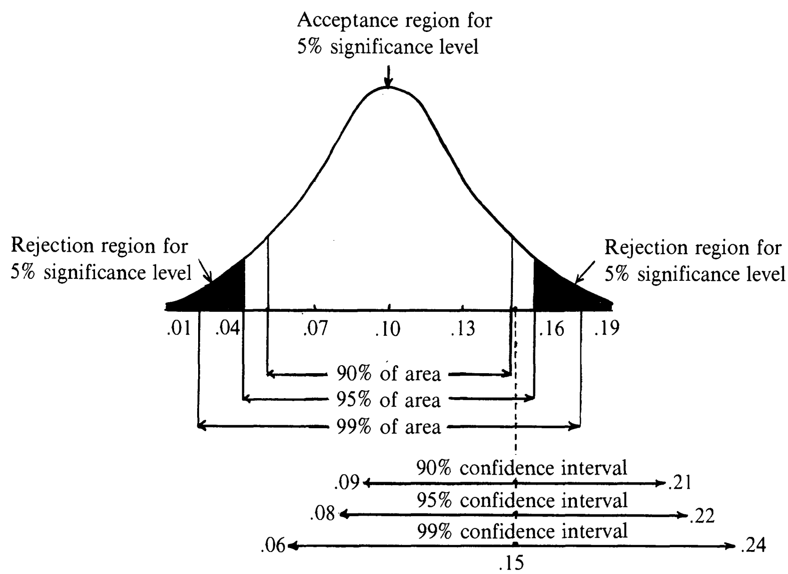


FIGURE 2. **Sampling distribution of $\hat{p}$ for samples of size 100 from a population in which $p=0.10$, and confidence intervals for p based on a sample in which $\hat{p}=0.15$.**

interval for p, we approximate the standard deviation $\sqrt{p(1-p)/n}$ of $\hat{p}$ by $\sqrt{\hat{p}(1-\hat{p})/n}$, and for this reason the equivalence here between confidence interval and the acceptance region of hypothesis testing is only approximate. If we're determined to eliminate the discrepancy, we can do so by utilizing the test statistic

$$Z=(\hat{p}-0.10)/\sqrt{\hat{p}(1-\hat{p})/n},$$

which is also asymptotically standard normal. Substitution of $\hat{p}=0.15$ and $n=100$ in this expression yields $Z=1.400$, implying that at the 10% level of significance our findings are consistent with the hypothesis that $p=0.10$. However, the customary practice is to use the test statistic as originally defined, so most statisticians would agree that our original decision to reject H_0 at the 10% significance level happens to be the right one after all.

The subjective side of confidence intervals

Are you a frequentist or a subjectivist? Your answer will influence the way you interpret confidence intervals, for the way we look at confidence intervals is inextricably linked to the way we look at probability itself. There is no unique interpretation of confidence intervals, just as there is no unique interpretation of probability. To say that there is only one viable view of probability is somewhat like insisting that space is Euclidean.

How is confidence in the 95% confidence interval, [0.08, 0.22], to be construed? Most authors [**3**, p. 106], [**5**, p. 157], [**10**, p. 245], [**14**, p. 164], [**15**, p. 176], [**16**, p. 303], [**17**, p. 204], [**18**, p. 275], [**19**, p. 298] advocate the strict interpretation that our confidence is in the *process* of taking random samples and obtaining intervals, and not in any specific interval, such as [0.08, 0.22] in the present case. These authors affirm that

$$P\left(\hat{p}-1.960\sqrt{\hat{p}(1-\hat{p})/n}\leqslant p\leqslant\hat{p}+1.960\sqrt{\hat{p}(1-\hat{p})/n}\right)=0.95,$$

but they deny that

$$P(0.08\leqslant p\leqslant 0.22)=0.95.$$

In other words, once the endpoints of the interval have been evaluated, the probability statement no longer holds. Why this subtle distinction? Because the unknown proportion p is either *in* the interval [0.08, 0.22] or *not* in that interval; there's really nothing random about it. The randomness lies in the process of generating the 95% confidence intervals. Inasmuch as 95% of the time (in the long run) this process yields intervals which contain p, it is to the process itself that we ascribe a probability of 0.95.

This strict interpretation of confidence interval springs from a **frequentist** interpretation of probability, in which the aspect of probability which is stressed is the tendency displayed by some chance devices to produce stable relative frequencies. This view identifies probability with the limit of a relative frequency: to say that p is the probability that an A is a B is simply to say that p is *the limit of the relative frequency of B's among A's* (as the number of observed A's is increased without bound) [**9**, p. 4]. For example, when flipping a coin a great many times, a frequentist would interpret $P(\text{heads})=1/2$ to mean that in the limit, half of the flips would be heads. Are you nodding in agreement with all this? If so, then you can call yourself a frequentist.

But the end result of interval estimation is a specific interval, such as the interval [0.08, 0.22] we've been considering. Exactly what meaning does *it* have? Not much, in the school of the frequentists, as we have seen. There is, though, another view of confidence intervals, one which is tied to a different view of probability. The theory of **subjective probability** has been created to enable one to talk about probabilities when the frequency viewpoint does not apply [**1**, p. 61]. This is often the case, since probability when used as a guide in life refers, not to the frequency with which a random variable will take a certain value, but to the likelihood that a certain *constant* will have a certain value [**8**, p. 25].

FIGURE 3. **P (heads) $=\frac{1}{2}$: what this means to you depends on whether you are a frequentist or a subjectivist.**

The basic thesis of the subjective theory of probability is that probability statements are statements concerning actual degrees of belief [**8**, p. 30]. In the example of flipping a coin, a subjectivist would interpret P (heads) $= .5$ to reflect the betting odds for heads, i.e., he would say there were even odds for heads vs. tails. Thus in the subjective theory of probability, we define the probability of an event as a number between 0 and 1 that reflects our personal assessment of the chance that the event will occur. Notice that this value is not uniquely determined, inasmuch as it depends on the inclination of the person whose degree of belief that probability represents. This nonuniqueness clearly differentiates subjective probabilities from classical probabilities, which derive either from the full understanding of the mechanics of a process such as dice shooting or from extensive empirical observation of relative frequencies [**21**, p. 214]. Aside from lacking the precision of classical probabilities, subjective probabilities are problematical in other ways.

In practice, how do we determine the subjective probability of an event? The easiest way is to compare events, determining relative likelihoods. For example, to find $P(E)$, the probability of event E, compare E with E^c, the complement of E. If we feel that E is about twice as likely to happen as E^c, then we fix $P(E)$ at $2/3$ [**1**, p. 62]. Put another way, our degree of belief in a given statement may be indicated by the odds at which we're willing to bet on its truth [**8**, p. 5]. If this approach appeals to you, consider yourself a subjectivist. Be aware, however, of the problems that are peculiar to subjective probabilities. In addition to the lack of precision caused by the varying degrees of belief different persons have in the same statement, subjective probabilities are prone to a certain fuzziness even when just one person is involved. This fuzziness is suggested by our own inconsistency when we're forced to compare events several times, especially if we don't realize we're making the same comparison or can't remember what our previous choice was [**13**, p. 373]. We can't let these probabilities become too fuzzy, however, because even subjective probabilities are bound by certain laws. For example, we have to guard against this type of irrational assignment of probabilities: $P(A)=1/3$, $P(B)=1/3$, P (either A or B occurring) $=3/4$ [**1**, p. 62].

We come now to the main point of this section of the paper. Certainly the subjective rather than frequency viewpoint of probability applies when we speak of the probability that there will be another frost this spring, or that the stock market will fall, or that the U.S. will retain the America's Cup. So too, does the subjective concept apply to the probability that the proportion of smokers in the U.S. is between 0.3 and 0.4, or the probability that the proportion of students in the college favoring stricter discipline is between 0.08 and 0.22. In keeping with this view we adopt a looser interpretation of confidence intervals. In short, we place our confidence in the specific interval as well as in the general process of generating the confidence intervals, by our assertion that

$$P(0.09 \leqslant p \leqslant 0.21) = 0.90,$$

$$P(0.08 \leqslant p \leqslant 0.22) = 0.95,$$

$$P(0.06 \leqslant p \leqslant 0.24) = 0.99.$$

Some authors [**2**, p. 254], [**4**, p. 262], [**11**, p. 237] espouse this loose interpretation of confidence intervals, but they do so with an apology for what they deem a lack of precision or even an abuse of language. To my way of thinking, this interpretation requires no apology. My proposal is that having noted with those of the frequentist persuasion that the process of generating 95% confidence intervals works 95% of the time, let us simply set 0.95 equal to the probability that a specific 95% confidence interval contains the population proportion p.

Demonstrations on confidence intervals through computer-simulated sampling

As suggested by the opening anecdote of this paper, students sometimes find these concepts hard to grasp. One way of reducing the abstractions to more concrete terms is to illustrate the meaning of confidence intervals through computer-simulated sampling.

Using this technique, you can demonstrate for yourself that the process of generating 95% confidence intervals yields intervals which, 95% of the time, contain the population proportion p, provided the assumptions underlying the process are satisfied (and if they are not satisfied, that the process fails). The BASIC program listed below will enable you to do just that. This program produces a large number m of simple random samples of size n from a population in which the relevant proportion p is $1/L$ and for each sample computes a $100(1-\alpha)\%$ confidence interval for p. Then the program computes the percentage of the m intervals that actually contain p. Notice that in the second line, the program directs you to "ENTER L, N, M, A, AND Z." Choose your population proportion p to be of the form $1/2, 1/3, 1/4, \ldots$, and then let L be the integer $1/p$. As indicated above, "N" is the sample size and "M" is the number of samples to be generated. For 90%, 95%, or 99% confidence intervals, let "A" equal 0.10, 0.05, 0.01, respectively, and let "Z" equal 1.645, 1.960, 2.576, respectively. Other combinations of "A" and "Z" are permissible provided, of course, that "A" equals the probability that the standard normal variable is greater than "Z."

```
100 REM- CONFIDENCE INTERVALS FOR P
110 INPUT "ENTER L, N, M, A, AND Z"; L,N,M,A,Z
120 RANDOMIZE: P = 1/L: B = 100*(1 - A):T = 0
130 DIM P(M), S(M): FOR J = 1 TO M: X = 0
140 FOR I = 1 TO N: R = INT(RND*L)
150 X = X - (R = 0): NEXT I
160 P(J) = X/N: S(J) = SQR(P(J)*(1 - P(J))/N)
170 T = T - ((ABS(P(J) - P)) < (Z*S(J))): NEXT J
180 C = 100*T/M
190 LPRINT "FOR L, N, M EQUAL TO ";L,N,M
200 LPRINT "% OF ";B;"% CONFIDENCE INTERVALS CONTAINING P: ";C
210 END
```

	L	N	M	A	% CONFIDENCE INTERVALS CONTAINING P
(1)	4	40	100	0.05	94
(2)	20	40	100	0.05	88
(3)	100	40	100	0.05	38
(4)	10	10	100	0.05	63
(5)	10	40	100	0.05	90
(6)	10	80	100	0.05	89
(7)	4	40	100	0.10	88
(8)	4	40	100	0.05	95
(9)	4	40	100	0.01	99

TABLE 1

The usefulness of this program lies in its ability to generate and analyze a large number of samples within a relatively short period of time. You can vary the parameters at will and see the effect immediately. TABLE 1 displays data obtained on a number of runs of this program. The number M of samples was 100 throughout, each of the other parameters was varied in turn. In rows (1) through (3) $P = 1/L$ was varied, in rows (4) through (6) n was varied, and in rows (7) through (9), α was varied.

Notice in TABLE 1 that only in two cases did the number of confidence intervals containing p fall far below the number expected. This happened in row (3), where the population proportion p is a small 0.01, and in row (4), where the sample size n is only 10. In both instances, np is less than the required 5: $np = 0.4$ for row (3) and $np = 1$ for row (4). Thus these experimental results are in agreement with the theory.

Other applications of this program will occur to the creative teacher. And to struggling students as well.

References

[1] J. O. Berger, Statistical Decision Theory, Springer-Verlag, New York, 1980.

[2] J. D. Braverman, Fundamentals of Business Statistics, Academic, New York, 1978.

[3] R. E. Chandler, The statistical concepts of confidence and significance, Statistical Issues, edited by R. E. Kirk, Brooks/Cole, Monterey, CA, 1972.

[4] B. Gelbaum and J. G. March, Mathematics for the Social and Behavioral Sciences, Saunders, Philadelphia, PA, 1969.

[5] D. G. Haack, Statistical Literacy: A Guide to Interpretation, Duxbury, N. Scituate, MA, 1979.

[6] I. Hacking, The Emergence of Probability, Cambridge, New York, 1975.

[7] P. G. Hoel and R. J. Jessen, Basic Statistics for Business and Economics, Wiley, New York, 1971.

[8] H. E. Kyburg, Jr., Probability and the Logic of Rational Belief, Wesleyan Univ. Press, Middletown, CT, 1961.

[9] H. E. Kyburg, Jr., and H. E. Smokler, Editors, Studies in Subjective Probability, Wiley, New York, 1964.

[10] L. Lapin, Statistics for Modern Business Decisions, Harcourt Brace Jovanovich, New York, 1978.

[11] H. J. Larson, Introduction to Probability Theory and Statistical Inference, Wiley, New York, 1968.

[12] S. G. Levy, Inferential Statistics in the Behavioral Sciences, Holt, Rinehart, and Winston, New York, 1968.

[13] R. D. Luce and H. Raiffa, Games and Decisions, Wiley, New York, 1957.

[14] E. Lukacs, Probability and Mathematical Statistics, Academic, New York, 1972.

[15] H. J. Malik and K. Mullen, Applied Statistics for Business and Economics, Addison-Wesley, Reading, MA, 1975.

[16] P. L. Meyer, Introductory Probability and Statistical Applications, Addison-Wesley, Reading, MA, 1970.

[17] R. L. Mills, Statistics for Applied Economics and Business, McGraw-Hill, New York, 1977.

[18] D. S. Moore, Statistics: Concepts and Controversies, Freeman, San Francisco, CA, 1979.

[19] F. Mosteller, R. E. K. Rourke, and G. B. Thomas, Probability with Statistical Applications, 2nd ed., Addison-Wesley, Reading, MA, 1973.

[20] M. G. Natrella, The relation between confidence intervals and tests of significance, Statistical Issues, edited by R. E. Kirk, Brooks/Cole, Monterey, CA, 1972.

[21] H. Schwarz, The use of subjective probability methods in estimating demand, Statistics: A Guide to the Unknown, edited by J. M. Tanur, Holden-Day, San Francisco, CA, 1972.

NOTES

Lessons from the Greeks and Computers

WILLIAM L. BRIGGS
Clarkson College of Technology
Potsdam, NY 13676

> There is a sweet and a bitter, a hot and a cold and according to convention there is color. In truth there are atoms and a void.
>
> —Democritus

> You cannot step into the same river twice, for fresh waters are ever flowing in upon you.
>
> —Heraclitus

Two very different modes of thought, reflected in the passages above, have pervaded mathematics since its early beginning. They have developed in parallel over the years, with varying degrees of independence, enhancing and opposing each other, uniting and dividing mathematics right up to the present day. With a few notable exceptions, the mathematicians of antiquity worked in the **discrete mode**. Using methods and arguments which were finite in nature, these early mathematicians were able to solve many classical problems of geometry (for example, determining the perimeter and area of plane polygonal regions or the surface area and volume of regular polyhedra). With the introduction of the calculus in the seventeenth century, mathematicians began to work in what could be called the **continuous mode**. They developed ways in which to deal with infinite sets (both countable and uncountable) and infinitesimal quantities; these led to solutions of more general problems (for example, finding the area and perimeter of plane regions with curved boundaries or summing an infinite set of numbers).

As we shall see, the two modes needn't be used in exclusion of each other; rather, they may be coordinated in very powerful ways. When the exact solution of a discrete problem is known, it may be used to generate a sequence of approximations to a related continuous problem. The "exact" solution of the continuous problem can then often be obtained as the limit of such a sequence of approximations. Although this approach can be exceedingly useful for analytic considerations, it often yields little *numerical* information about the answer. Today, we have come full circle—with the computer dominating the field of mathematical computation, we have returned to an emphasis on discrete methods. We have learned lessons from the Greeks, and improved on their techniques.

Probably the most distinguished and widely considered problem of early mathematics was the calculation of π. We will look at the way in which ancient mathematicians, particularly the Greeks, approached this problem and the solutions which they found. Their methods will then be compared with the methods which are used today to solve similar problems on a computer. Both the similarities and improvements will be discussed.

The history of π (see [**2**], [**8**], [**10**], [**11**]) is long and fascinating and deserves at least a brief elaboration. As pointed out in [**2**], early stick and rope experiments would have sufficed to approximate π by the inequality

$$3\frac{1}{8} < \pi < 3\frac{1}{7}.$$

We know that by 2000 B.C. the Egyptians assumed a value of $256/81$ ($=3.1604938$) for π in using the formula $A=(8d/9)^2$ for the area of a circle with diameter d. About the same time, the Babylonians gave a value of $3\frac{1}{8}$ for π by approximating the perimeter of a circle by an inscribed hexagon. In the first century, the Chinese mathematician Ch'ang Hong was one of the first to use $\sqrt{10}$ as an approximation to π. In the third century Wang Fan offered $142/45$ ($=3.1\bar{5}$) as an approximation and by 500 A.D. Tsu Ch'ung-chih replaced a current approximation of $22/7$ by $355/113$ ($=3.14159292\ldots$). Approximations of π appeared in India also. Values of 3 and $\sqrt{10}$ were variously used in problems of mensuration. Mahavira used the formula $\frac{9}{10}\cdot\frac{9}{2}(\frac{1}{2}d)^3$ for the volume of a sphere with diameter d, giving a value for π of 3.0375. In about 500 A.D. Aryabhatta used $62832/20000$ ($=3.1416$) for π to compute the circumference of a circle with a diameter of 20,000.

The documentation of the calculation of π becomes much better with the Greek mathematicians. Euclid (~ 300 B.C.) proved that circles are one to another as the squares of their diameters; that is, the ratio of the area of a circle to the square of its diameter is constant for all circles. It remained only to find this constant. In his treatise *Measurement of a Circle*, Archimedes (287–212 B.C.) was able to show that the area of a circle is equal to the area of a right triangle whose sides have length equal to the radius and circumference of the circle. The area of the triangle could be calculated. Therefore for Archimedes the problem of finding the area of a circle (and hence π) became the problem of calculating the circumference of a circle with a given radius. His method was to approximate the circle by regular inscribed and circumscribed polygons. As the number of sides of the polygons is increased, their perimeter more nearly approximates the circumference of the circle. Archimedes' famous estimate of π based on polygons with 96 sides is given by

$$3\frac{10}{71} < \pi < 3\frac{1}{7}. \tag{1}$$

He most likely began with inscribed and circumscribed hexagons and doubled the number of sides four times ($n=12, 24, 48, 96$) [7].

A possible reconstruction of Archimedes' argument is illustrated in FIGURE 1. (A slightly more involved development is given in [**12**].) Given a circle, let $2s$ and $2t$ be the length of the sides of a circumscribed and an inscribed n-gon, respectively. On doubling the number of sides, let $2s'$ and $2t'$ be the length of the sides of the corresponding $2n$-gons. We would like to relate s' and t' to s and t. Since triangles ABC and AFD are similar, it follows that

$$\frac{s}{t}=\frac{s-s'}{s'},$$

so

$$s'=\frac{st}{s+t}. \tag{2}$$

The similarity of triangles FBC and DEC gives

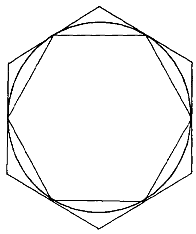
$$\frac{2t'}{t}=\frac{s'}{t'},$$

so

$$t'=\sqrt{\frac{s't}{2}}. \tag{3}$$

Let S_n be the perimeter of the circumscribed n-gon, and T_n the perimeter of the inscribed n-gon; then $S_n=2ns$ and $T_n=2nt$. From (2) and (3), it follows that

$$S_{2n}=\frac{2S_nT_n}{S_n+T_n} \quad \text{and} \quad T_{2n}=\sqrt{T_nS_{2n}}. \tag{4}$$



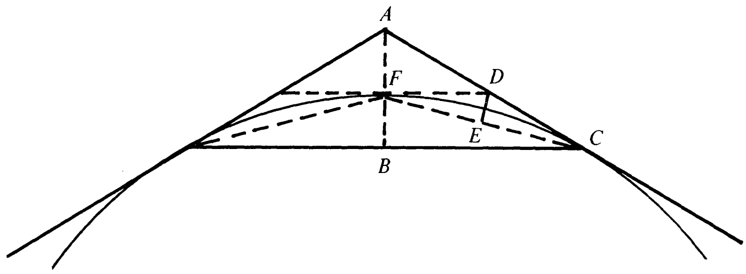


FIGURE 1. **Doubling the number of sides of n-gons circumscribed and inscribed in a circle.**

The equations in (4) provide a method of systematically calculating the perimeter of circumscribed and inscribed n-gons with n doubling at each step. (The reader may check that, beginning with $n = 6$, Archimedes' inequality (1) can be obtained after four iterations.)

The approximation to π just discussed resulted from approximating the circumference of the circle. We note that the Greeks obtained other approximations to π by considering another large class of problems, namely, computing areas and volumes of polygons and polyhedra. Archimedes produced results on the volume and surface area of the sphere and the cylinder; he knew of the relation between the area of a circle and the area of an ellipse, and in one celebrated theorem he gave an expression for the area of a segment of a parabola. (This latter result in particular relied on the so-called method exhaustion and represents an excursion into the realm of the continuous. For this reason Archimedes is often credited with having the earliest insight into the fundamental ideas of calculus.)

How might the problem of calculating π be solved today with the use of computers? This involves the subject of numerical analysis. Although computers are powerful and fast they are still entirely finite; any solution that a computer gives to a continuous problem is necessarily an approximation (unless we are lucky!). The job of numerical analysis is to make these approximations as good as possible.

Let us return to Archimedes' calculation of π, and assume for simplicity that our approximating n-gons are circumscribed and inscribed in a circle with diameter 1. The number π is then the circumference of the circle. To begin the iteration of values for T_n and S_n using (4), we need an initial value n to start the calculation. For $n = 4$, we have $T_4 = 2\sqrt{2}$ and $S_4 = 4$; the perimeters for successive polygons using (4) for nine iterations are shown in TABLE 1.

n	S_n	T_n
4	4.000000000	$2\sqrt{2}$
8	3.313708496	3.061467460
16	3.182597878	3.121445153
32	3.151724908	3.136548491
64	3.144118386	3.140331158
128	3.142223630	3.141277252
.		
.		
.		
1024	3.141602512	3.141587726
2048	3.141595118	3.141591422

TABLE 1

From the geometry of the problem, we know that $T_n < \pi < S_n$ for all n. Additionally, as n increases, T_n increases and S_n decreases to approach π more and more closely. Since T_n and S_n bound π below and above, it follows that whenever T_n and S_n agree to p digits, then π also has those same p digits in its representation. In the calculation shown, T_{2048} and S_{2048} have six digits in agreement and we may conclude that $\pi = 3.14159\ldots$. More iterations will determine more digits.

With a computer, this algorithm (or computational procedure) may be carried out to very large values of n and π may be determined with more accuracy than Archimedes could have ever hoped. But at the same time both the computer and Archimedes are constrained by two facts: the discreteness of the process, and the accuracy of the calculations which produce more digits. No matter how many digits of π are determined with certainty, there are always more digits that remain undetermined. The sequences S_n and T_n may agree to all calculated digits for some n—but we cannot conclude from this that π *equals* their common value (in fact, we *know* it doesn't!). In addition, computer round-off or other computational errors may prevent accuracy of calculation beyond a certain number of decimal places.

We now consider an improvement on Archimedes' calculation. Although a computer cannot make the transition from the discrete to the continuous, it can be made to approach the transition "more quickly." This idea is the basis of *acceleration* techniques.

Consider the sequence $\{S_4, S_8, S_{16}, \ldots\}$ which approaches π and relabel it $\{p_1, p_2, p_3, \ldots\}$. The first acceleration technique which we will consider is called the **Aitken δ^2 process** [**1**], [**4**]. The error in a particular approximation p_n is simply $p_n - \pi$. If there exists a constant λ such that

$$p_n - \pi \approx \lambda(p_{n-1} - \pi)$$

as n becomes large, then the sequence $\{p_n\}$ is said to converge linearly and the following argument applies. We observe that

$$\begin{aligned} p_n - \pi &\approx \lambda(p_{n-1} - \pi), \\ p_{n+1} - \pi &\approx \lambda(p_n - \pi). \end{aligned}$$

Eliminating λ and solving for the desired limit π gives

$$\pi \approx p_{n-1} - \frac{(p_n - p_{n-1})^2}{p_{n+1} - 2p_n + p_{n-1}}. \tag{5}$$

The expression on the right-hand side of (5) defines the elements p'_{n+1} of a new sequence, called an *accelerated* sequence. For example, for our sequence $p_n = S_{2^{n+1}}$ shown in TABLE 1, the first term of the accelerated sequence derived from it is

$$p'_3 = p_1 - \frac{(p_2 - p_1)^2}{p_3 - 2p_2 + p_1};$$

if we substitute the values of S_4, S_8, and S_{16}, in TABLE 1, for p_1, p_2, and p_3, we obtain $p'_3 = 3.151635007$. The advantage to acceleration is this: if the original sequence converges linearly, then the accelerated sequence converges more rapidly than the original sequence [**5**]. After obtaining the accelerated sequence from the original sequence, acceleration may then be performed in the new sequence.

In our example, the sequence of approximations $\{S_4, S_8, S_{16}, \ldots\}$ can be shown to converge linearly to π. TABLE 2 shows the result of applying two accelerations to this sequence.

From this table, we see that S''_{64}, the first term of the sequence after two accelerations, has more correct digits of π than S_{2048}, the tenth term of the original sequence.

The Greek idea of approximating π by computing the area of a plane region leads to the problem familiar to students: find the area under a curve or, equivalently, evaluate a definite

n	S_n	S'_n	S''_n
4	4		
8	3.313708496		
16	3.182597878	3.151635007	
32	3.151724908	3.142216103	
64	3.144118386	3.141631579	3.141592904
128	3.142223636	3.141595085	3.141592655

TABLE 2

integral. There are many definite integrals whose value is π; one of the simplest is

$$4\int_0^1 \frac{dx}{1+x^2} = \pi. \tag{6}$$

In this form, little numerical information about the value of π is revealed. By its very definition, however, the definite integral is a limit of a sequence of finite sums which can be calculated, and so once again the computer can give approximations to the limit π. Many of these approximations are based on the same ideas which the Greeks used: the plane region under the curve is approximated by n smaller polygonal regions whose area can be found, the regions chosen so that as n increases these approximations converge to the area under the curve. Two familiar methods of approximation based on this idea use rectangles and trapezoids: the Midpoint and Trapezoid Rules. Let $M(n)$ and $T(n)$ denote the Midpoint and Trapezoid Rule approximations to the integral (6) where the interval $[0,1]$ is partitioned into n equal parts. TABLE 3 shows calculations for $n=2^k$, $k=0, 1, 2, 3, 4$.

n	$M(n)$	$T(n)$
1	3.20	3.0
2	3.1623529	3.10
4	3.1468005	3.1311765
8	3.1428947	3.1389884
16		3.1409416

TABLE 3

Computation with both methods can be facilitated using the relation $T(2n)=\frac{1}{2}(T(n)+M(n))$. It is easy to see that $T(n)<\pi<M(n)$; both sequences converge as n is increased. But neither approximation is that good, and both sequences converge rather slowly. Acceleration can hasten convergence and approach the continuous limit more quickly.

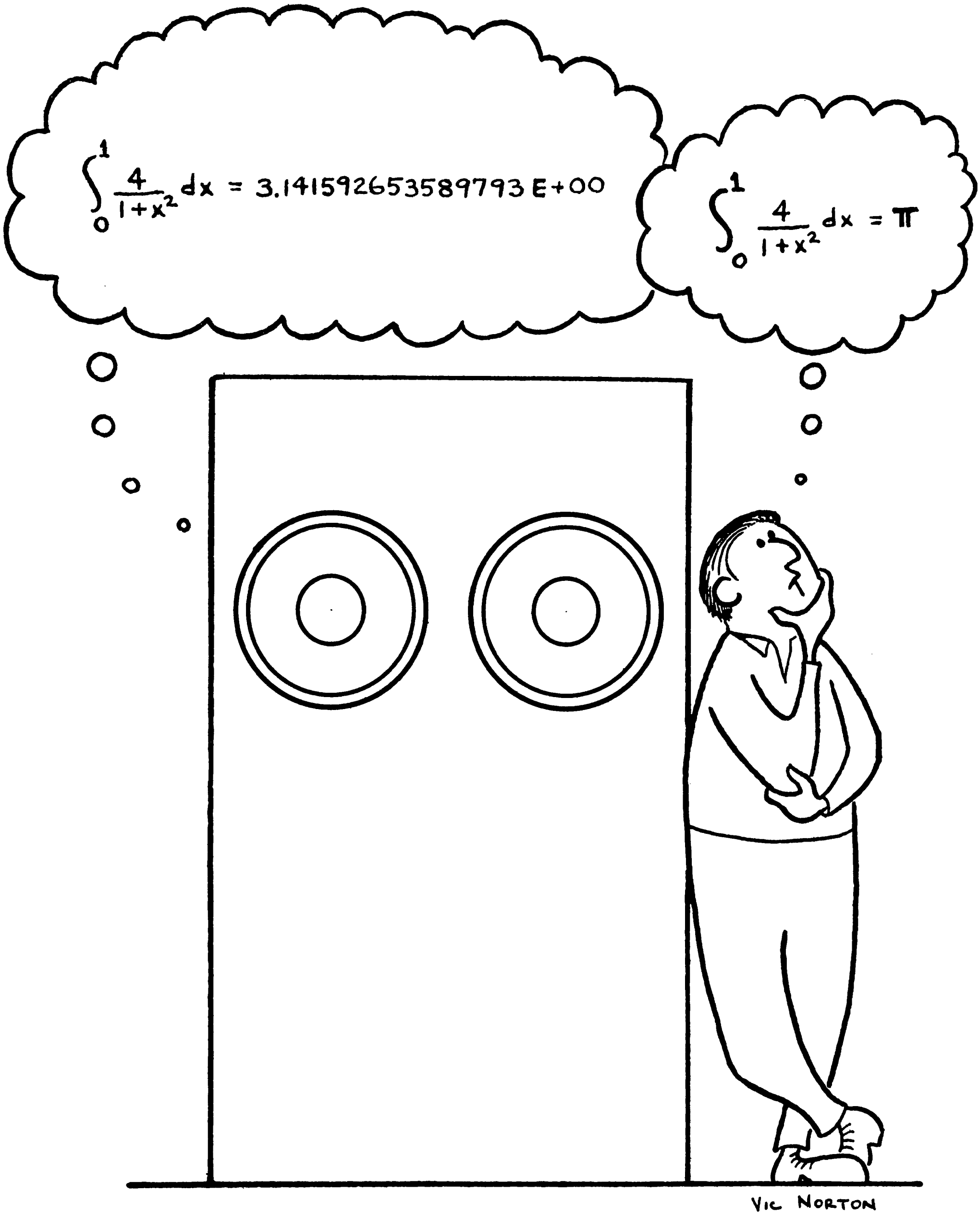
Extrapolation, or **Romberg Integration** [3], [**9**] is a different acceleration technique that can be applied to our sequences approximating π. We will demonstrate its application to the sequence $T(n)$. A well-known result (the Euler-Maclaurin Theorem) can be used to show that the error in a Trapezoid Rule approximation to a particular definite integral I has the form

$$I-T(n)=c_1n^{-2}+c_2n^{-4}+\cdots+c_nn^{-2n}+\cdots. \tag{7}$$

The constant coefficients c_i involve higher derivatives of the integrand; therefore the method depends on the integrand being sufficiently differentiable. Whenever an error term has an expansion as in (7) for n large, the following argument can be applied.

The aim now is to devise a rule whose error is proportional, not to n^{-2} as in the Trapezoid Rule, but proportional to n^{-k} for some $k>2$. If we substitute $2n$ for n in (7) and multiply by 4, we obtain

$$4(I-T(2n))=c_1n^{-2}+c_22^{-2}n^{-4}+c_32^{-4}n^{-6}+\cdots. \tag{8}$$



Now subtract (8) from (7) and solve for I:

$$I = \frac{4T(n) - T(2n)}{3} + c_2' n^{-4} + c_3' n^{-6}, \tag{9}$$

where c_i' are constants. Denote

$$T_1(n) = \frac{4T(n) - T(2n)}{3};$$

then $T_1(n)$ is a new approximation to I. Furthermore, the error in $I - T_1(n)$ is proportional to n^{-4} rather than n^{-2}. (If the width of the subintervals is halved, errors decrease by $1/16$.) This new quadrature rule can be shown to be the well-known Simpson's Rule.

However the process needn't stop here. We can repeat the process above and continue to eliminate successive terms in the error expansion to generate higher order approximations. (Thus, after computing $T_1(2n)$ and $T_1(n)$, the n^{-4} term can be eliminated to give an approximation with an error proportional to n^{-6}.) In general, it can be shown that the approximation

$$T_k(n)=\frac{4^k T_{k-1}(n)-T_{k-1}(2n)}{4^k-1}$$

has error proportional to n^{-2k-2} for $k \geqslant 1$ (define $T_0(n)=T(n)$).

Extrapolation proves to be much more efficient than simply taking larger values of n in the Trapezoid Rule. Extrapolation requires no new function evaluations and leads to more accurate approximations at the same time. If we apply the technique to our sequence $T(n)$ which approximates π, we obtain the values in TABLE 4.

n	$T_0(n)$	$T_1(n)$	$T_2(n)$
1	3.0		
2	3.10	3.133333	
4	3.1311765	3.1415684	3.1421174
8	3.1389884	3.1415924	3.1415940
16	3.1409416	3.1415928	3.1415928

TABLE 4

By taking n no larger than before ($n=16$), but now extrapolating twice, we find π correct to six digits ($T_2(16)=3.14159$). To have obtained this accuracy with the Trapezoid Rule alone would have required $n^{-1}\approx .0008$, resulting in 1250 evaluations of the integrand in (6) rather than just the 16 evaluations needed with extrapolations.

We have learned lessons from the Greeks. The spirit and technique of their approximation methods is alive and well. But we have developed some new techniques to obtain accurate approximations more quickly—we have acceleration techniques and computers to calculate for us.

References

[1] A. C. Aitken, Studies in practical mathematics, I and II, Proc. Royal Soc. Edinburgh, 57(1936, 1937) 172, 269.

[2] Petr Beckmann, A History of Pi, Golem Press, Boulder, CO, 1970.

[3] W. L. Briggs, Romberg Integration and Extrapolation Methods, Didactic Programming, fall 1979.

[4] R. Burden, J. Faires, A. Reynolds, Numerical Analysis, Prindle, Weber and Schmidt, 1978.

[5] G. Dahlquist, A. Bjorck, Numerical Methods, Prentice-Hall, 1974.

[6] P. Davis and P. Rabinowitz, Methods of Numerical Integration, Academic Press, New York, 1975.

[7] T. L. Heath, A Manual of Greek Mathematics, Oxford University Press, New York, 1931.

[8] M. Kline, Mathematical Thought from Ancient to Modern Times, Oxford University Press, New York, 1972.

[9] W. Romberg, Vereinfachte Numerische Integration, Norske Vid. Selsk. Forh. (Trondheim), 28 (1955) 30.

[10] D. E. Smith, History of Mathematics, vol. 1, Dover, New York, 1951.

[11] D. J. Struik, A Concise History of Mathematics, Dover, New York, 1967.

[12] D. W. TeSelle, Pi, polygons and a computer, Math. Teacher, 63 (1970) 128–132.

[13] H. Von Baravalle, The number π, Math. Teacher, 60 (1967) 479–487.

What Goes Up Must Come Down; Will Air Resistance Make It Return Sooner, or Later?

JOHN LEKNER
Victoria University
Wellington, New Zealand

A ball thrown straight up with speed v_i would, in the absence of air, return in time $2v_i/g$. Air resistance, or drag, will influence the return time in two ways: the maximum height reached is less than the zero-drag height $v_i^2/2g$, and the speed at any height z is less than the zero-drag speed. (These statements follow from the energy equation $\frac{1}{2}mv_i^2 = \frac{1}{2}mv(z)^2 + mgz + W$, where m is the mass of the ball, and W is the (positive) work done against air resistance. The speed is zero at the top of the trajectory, so $z_{\max} < v_i^2/2g$; and at any z, $v(z) < \sqrt{v_i^2 - 2gz}$. Note that the energy conservation equation is not an additional physical principle: it follows from the equation of motion on multiplying by v and integrating.) Thus with air resistance, the ball has a *shorter distance* to travel, but at a *slower speed*. Which effect wins?

Let $f(v)$ be the deceleration due to the drag force. The equation of motion then reads $dv/dt = -g - f(v)$ on the way up, and $dv/dt = g - f(v)$ on the way down (it is convenient to deal with speeds rather than velocities in this context). We will assume that $f(v)$ has the property that there is just one speed at which the gravitational and drag forces are in balance. This defines the *terminal speed* v_t: $f(v_t) = g$. The terminal speed is a natural scaling parameter for this problem. Let $u = v/v_t$ and $\phi(u) = f(v)/f(v_t) = f(v)/g$. Then by integrating dt (obtained from the equation of motion) we find the time to go up to maximum height is

$$t_{\text{up}} = \int_0^{v_i} \frac{dv}{g + f(v)} = \frac{v_t}{g}\int_0^{u_i} \frac{du}{1+\phi(u)}, \tag{1}$$

and the time to come down is

$$t_{\text{down}} = \int_0^{v_f} \frac{dv}{g - f(v)} = \frac{v_t}{g}\int_0^{u_f} \frac{du}{1-\phi(u)}. \tag{2}$$

The speed on impact, v_f, is determined by the condition that the distance travelled on the way up is the same as that travelled on the way down. These are given by integrating $v\,dt$; we find u_f is determined by

$$\int_0^{u_i} \frac{u\,du}{1+\phi(u)} = \int_0^{u_f} \frac{u\,du}{1-\phi(u)}. \tag{3}$$

We are interested in the ratio τ of the return time to the zero-drag return time $2v_i/g$. From (1) and (2),

$$\tau = \frac{t_{\text{up}} + t_{\text{down}}}{2v_i/g} = \frac{1}{2u_i}\left[\int_0^{u_i} \frac{du}{1+\phi} + \int_0^{u_f} \frac{du}{1-\phi}\right]. \tag{4}$$

Physically, $f(v)$ must go to zero as v goes to zero. Thus Φ, the maximum value of $\phi(u)$, can be made arbitrarily small compared to unity when the initial speed v_i is chosen sufficiently small compared to the terminal speed v_t (u_i sufficiently small). We can therefore expand $[1 \pm \phi(u)]^{-1}$ in (3) and (4), to find

$$\frac{u_f}{u_i} = 1 - \frac{2}{u_i^2}\int_0^{u_i} u\phi\,du + O(\Phi^2)$$

$$\tau = 1 - \frac{1}{u_i^2}\int_0^{u_i} u\phi\,du + O(\Phi^2). \tag{5}$$

Thus any physically reasonable form of drag will make the ball return sooner, provided the launch speed is small compared to the terminal speed.

Wind tunnel experiments [**1**] on spheres show that the drag force is (approximately) proportional to v^2 in the Reynolds number range $10^3 \leqslant R \leqslant 10^5$. This covers the range of practical interest, provided the launch speeds are kept moderate (a sphere of diameter 1.5 cm and speed 10^3 cm/s has $R \cong 10^4$ in air). For $f = kv^2$ ($\phi = u^2$) we find from (3) and (4) that

$$u_f = \frac{u_i}{\sqrt{1+u_i^2}} \tag{6}$$

and

$$\tau = \left(\arctan u_i + \operatorname{arctanh} u_f\right)/2u_i. \tag{7}$$

The numerator $N(u) = \arctan u + \operatorname{arctanh}\left(u/\sqrt{1+u^2}\right)$ has slope $dN/du = (1+u^2)^{-1} + (1+u^2)^{-1/2}$, which is less than 2 for nonzero u. Thus $N(u_i)$ increases more slowly than $2u_i$, the leading term in its Taylor expansion about $u_i = 0$. It follows that, for a v^2 drag, τ is always less than unity, no matter what the initial speed.

Could this result be true for an arbitrary (nonnegative) drag $f(v)$? Let's try a few more examples. When f is linear in v (Stokes' law), we find the attractive result

$$\tau = \tfrac{1}{2}\left(1 + \frac{u_f}{u_i}\right)$$

or

$$t_{\text{up}} + t_{\text{down}} = \frac{v_i + v_f}{g}. \tag{8}$$

Since v_f is always less than v_i, we again have the return time being shortened by air resistance, irrespective of the initial speed.

So far, all has indicated a shorter return time. Now consider some fractional powers. First suppose $f(v) \sim v^{1/2}$. Setting $u = w^2$, we find that u_f is determined by an interesting transcendental equation

$$\frac{1}{3}w_i^3 - \frac{1}{2}w_i^2 + w_i - \log(1+w_i) = -\frac{1}{3}w_f^3 - \frac{1}{2}w_f^2 - w_f - \log(1-w_f), \tag{9}$$

and that the ratio of return time to zero-drag return time is

$$\tau = w_i^{-2}\{w_i - \log(1+w_i) - w_f - \log(1-w_f)\}. \tag{10}$$

For $u_i \gg 1$ we find $\tau \to \frac{1}{3}u_i^{1/2}$, larger than unity.

Next, suppose $f(v) \sim v^{2/3}$. Setting $u = y^3$ we find

$$\frac{1}{2}y_i^4 - y_i^2 + \log(1+y_i^2) = -\frac{1}{2}y_f^4 - y_f^2 - \log(1-y_f^2) \tag{11}$$

and

$$\tau = \frac{3}{2y_i^3}\{y_i - \arctan y_i + \operatorname{arctanh} y_f - y_f\}. \tag{12}$$

For $u_i \gg 1$, $\tau \to \frac{3}{8}u_i^{1/3}$, again larger than unity.

The above results suggest to me that there is a cross-over at the linear force law:

CONJECTURE. *For powers p in $f(v) = kv^p$, $p \geqslant 1$ gives a return time which is always shorter than the zero-drag return time $2v_i/g$. For $p < 1$, the return time is shorter for small initial speeds, but eventually becomes longer than $2v_i/g$ as v_i increases. The closer p is to 1, the higher the ratio of the initial speed to the terminal speed before this happens.*

We have determined $\tau(u_i)$ for only four values of p: 2, 1, 1/2, 2/3. Students may enjoy some of the following projects in analysis and numerical methods:

(a) plotting τ versus u_i for these four values of p;

(b) finding other values of p for which the integral equation (3) for u_f is reducible to a transcendental equation, and plotting $\tau(u_i)$ for these;

(c) a class exercise in which different values of $p < 1$ are assigned to students or student groups, and each is asked to find the u_i for which $\tau = 1$.

This problem originated from a first-year physics question set by Tim Shirtcliffe. I am grateful to him and to John Harper and Graeme Wake for helpful comments.

Reference

[1] G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge, 1967, p. 341.

A Method of Duplicating the Cube

Edward V. Graef
1814 Kent Road
Pittsburgh, PA 15241

V. C. Harris
San Diego State University
San Diego, CA 92182

The old problem of duplicating the cube—that is, of constructing a cube with volume twice that of a given cube—was solved geometrically in several ways by the ancient Greek mathematicians (see Eves [**1**] for a summary). It is the purpose of this note to show how analytic geometry can be used to construct two curves which will give one more solution to the problem.

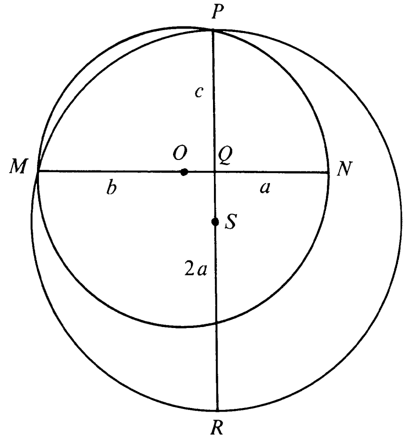


Figure 1

We have determined $\tau(u_i)$ for only four values of p: 2, 1, 1/2, 2/3. Students may enjoy some of the following projects in analysis and numerical methods:

(a) plotting τ versus u_i for these four values of p;

(b) finding other values of p for which the integral equation (3) for u_f is reducible to a transcendental equation, and plotting $\tau(u_i)$ for these;

(c) a class exercise in which different values of $p < 1$ are assigned to students or student groups, and each is asked to find the u_i for which $\tau = 1$.

This problem originated from a first-year physics question set by Tim Shirtcliffe. I am grateful to him and to John Harper and Graeme Wake for helpful comments.

Reference

[1] G. K. Batchelor, An Introduction to Fluid Dynamics, Cambridge, 1967, p. 341.

A Method of Duplicating the Cube

EDWARD V. GRAEF
1814 Kent Road
Pittsburgh, PA 15241

V. C. HARRIS
San Diego State University
San Diego, CA 92182

The old problem of duplicating the cube—that is, of constructing a cube with volume twice that of a given cube—was solved geometrically in several ways by the ancient Greek mathematicians (see Eves [**1**] for a summary). It is the purpose of this note to show how analytic geometry can be used to construct two curves which will give one more solution to the problem.

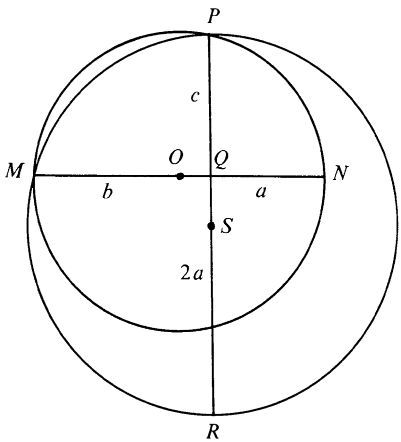


FIGURE 1

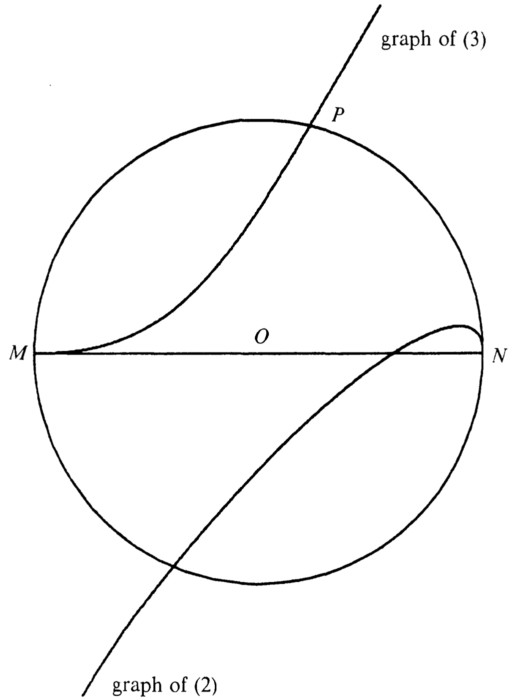


FIGURE 2

Given a cube whose edge has length c, if numbers a and b can be found such that

$$\frac{2a}{b}=\frac{b}{c}=\frac{c}{a}, \tag{1}$$

then the problem is solved, because $b^2=2ac$ and $c^2=ab$ imply $b^3=2abc$ and $c^3=abc$, so $b^3=2c^3$. The relations (1) can be thought of as relations between segments of two perpendicular chords of intersecting circles (see FIGURE 1). The problem is, given c and circle MPN with center at the origin O and radius r, to determine circle MPR with center $S(x,y)$ and radius $a+(c/2)$.

First we obtain the locus of the centers of all circles such that $|QR|=2|QN|$, without requiring that they also pass through M. The y-coordinate of R is $-2a=-2(r-x)$, and the y-coordinate of P is $(r^2-x^2)^{1/2}$, so the y-coordinate of S, their midpoint, is

$$y=\frac{1}{2}\Big((r^2-x^2)^{1/2}-2(r-x)\Big). \tag{2}$$

Next we take circles with centers on the graph of (2) which in addition pass through $M(-r,0)$ and obtain the locus of the upper endpoints of their vertical diameters. If $T(x,Y)$ is a point on that locus then, since $|ST|=|SM|$,

$$Y-y=\Big((r+x)^2+y^2\Big)^{1/2}, \tag{3}$$

where y is as in (2). FIGURE 2 shows the graphs of equations (2) and (3). Where the graph of (3) intersects the circle—that is, where $Y=(r^2-x^2)^{1/2}$— determines the point P and hence S. This duplicates the cube, because (3) reduces to $(r+x)^3=2((r^2-x^2)^{1/2})^3$, or $b^3=2c^3$.

The authors wish to thank the referee for helpful suggestions.

Reference

[1] Howard Eves, An Introduction to the History of Mathematics, Holt, Reinhart and Winston, New York, 1964.

The Pure Theory of Elevators

A WUFFLE
School of Social Sciences
University of California, Irvine
Irvine, CA 92717

Let us consider three puzzles which will turn out to be mathematically very similar.

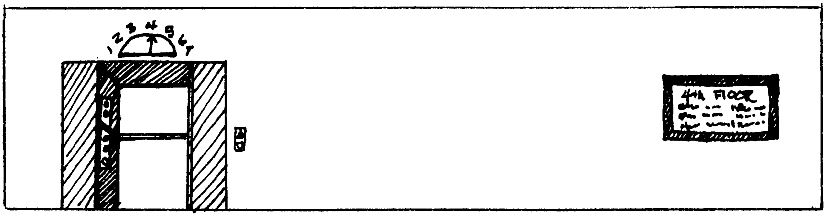
PUZZLE 1. It has often been noted that, unless you are waiting at either the bottom floor or at the top floor of a building, the next elevator is almost always going in the *wrong* direction. (See e.g., [**1**], 10–11.) On the other hand it seems reasonable, since what goes down must come up (the author is indebted to Miss Shari Shreiber for pointing this out to him), and conversely, that it ought to be just as easy (i.e., take just as long) to get an elevator going up as one going down. How can both of these things be true?

PUZZLE 2. A New Yorker has two girlfriends, one who lives uptown and one downtown, each of whom he likes equally. He goes (as the whim takes him) to the subway station and takes whichever train (uptown or downtown) comes along first. Trains run equally often in both directions. Despite his equal attachment to the two women he ends up seeing one considerably more often than the other. How can this be? (I learned of this puzzle through my colleague Louis Narens. See [**2**].)

PUZZLE 3. ([**1**], 59–60)

> In a small midwestern town there lived a retired railroad engineer named William Johnson. The main line on which he had worked for so many years passed through the town. Mr. Johnson suffered from insomnia and would often wake up at any odd hour of the night and be unable to fall asleep again. He found it helpful, in such cases, to take a walk along the deserted streets of the town, and his way always led him to the railroad crossing. He would stand there thoughtfully watching the track until a train thundered by through the dead of night. The sight always cheered the old railroad man, and he would walk back home with a good chance of falling asleep.
>
> After a while he made a curious observation; it seemed to him that most of the trains he saw at the crossing were traveling eastward, and only a few were going west. Knowing very well that this line was carrying equal numbers of eastbound and westbound trains, and that they alternated regularly, he decided at first that he must have been mistaken in this reckoning. To make sure, he got a little notebook, and began putting down "E" or "W", depending on which way the first train to pass was traveling. At the end of a week, there were five "E's" and only two "W's" and the observations of the next week gave essentially the same proportion. Could it be that he always woke up at the same hour of night, mostly before the passage of eastbound trains?
>
> Being puzzled by this situation, he decided to undertake a rigorous statistical study of the problem, extending it also to the daytime. He asked a friend to make a long list of arbitrary times such as 9:35 a.m., 12:00 noon, 3:07 p.m., and so on, and he went to the railroad crossing punctually at these times to see which train would come first. However, the result was the same as before. Out of one hundred trains he met, about seventy-five were going east and only twenty-five west. In despair, he called the depot in the nearest big city to find whether some of the westbound trains had been rerouted through another line, but this was not the case. He was, in fact, assured that the trains were running exactly on schedule, and that equal numbers of trains daily were going each way. This mystery brought him to such despair that he became completely unable to sleep and was a very sick man.

Let us look at Puzzle 1 first. Consider a building with N floors and r elevators. To simplify, let us begin with the special case $r=1$. At any moment the elevator can be in any one of $2(N-1)$ states, which we may denote $1\uparrow, 2\uparrow, 2\downarrow, 3\uparrow, 3\downarrow, \ldots, (N-1)\uparrow, (N-1)\downarrow, N\downarrow$. The arrow indicates the direction in which the elevator is going after it stops on the ith floor. Of course, at the top floor [bottom floor] the elevator must always next go down [up]. Let the time between two floors



be t. We shall assume that t incorporates some fixed (negligible) waiting time per floor. Consider a high traffic case where, for simplicity, we assume the elevator stops at each floor. (An anonymous referee has pointed out that orthodox Jews, who can't "operate" machinery on the Sabbath and other religious holidays, will sometimes set elevators so that they will automatically stop at every floor for a fixed amount of time. These are called *Shabbos elevators*.) If you are at floor k, what is the probability that the next elevator will be going up? Let us denote this probability as $p(k\uparrow)$. For $1<k<N$, the next elevator will be going up whenever the elevator is in one of the states $1\uparrow, 2\uparrow, \ldots, (k-1)\uparrow$ or in one of the states $2\downarrow, 3\downarrow, \ldots, k\downarrow$. There are $k-1+k-1$ $(=2k-2)$ such states. We may hypothesize that all $2(N-1)$ possible states are equally likely. For $1<k<N$,

$$p(k\uparrow)=\frac{2(k-1)}{2(N-1)}=\frac{k-1}{N-1} \tag{1}$$

and

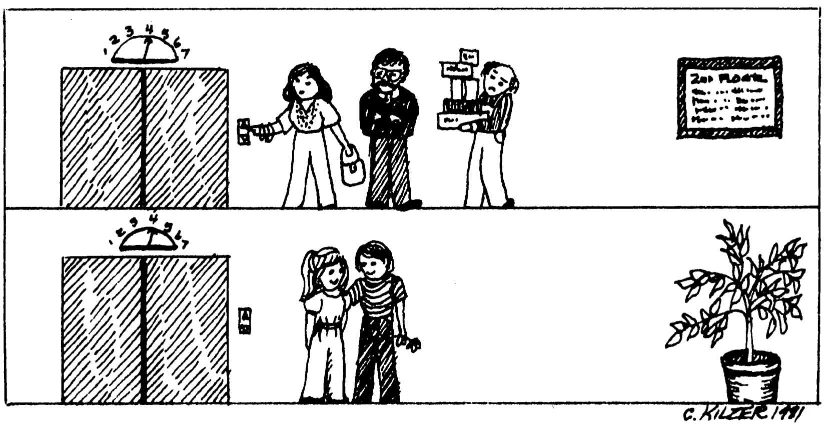
$$p(k\downarrow)=\frac{N-k}{N-1}=1-p(k\uparrow). \tag{2}$$

Of course $p(1\uparrow)=1$ and $p(N\downarrow)=1$. *Thus, for example, on the second floor of Social Sciences Tower at the University of California, Irvine* ($N=7$), *when one of the two elevators is* (*as happens quite often*) *out of order, the probability that the next elevator is going up is only* $1/6$.

Now, however, let us look at a different though related question. If we are standing on floor k, how long must we wait, on average, till the next up elevator? Consider the following chain:

$$k\uparrow, (k+1)\uparrow, (k+2)\uparrow, \ldots, (N-1)\uparrow, N, (N-1)\downarrow,$$
$$(N-2)\downarrow, (N-3)\downarrow, \ldots, 1\uparrow, 2\uparrow, 3\uparrow, \ldots, (k-1)\uparrow.$$

There will, of course, be exactly $2(N-1)$ states in this chain. If we are standing on floor k, the waiting time for an up elevator if the elevator is in state one of this chain is either 0 or $(2N-2)t$. We shall assume it is $(2N-2)t$, i.e., you just missed the elevator. The waiting time for an up elevator if the elevator is in state two of this chain is $(2N-3)t$. The waiting time for an up elevator if the elevator is in state j of this chain, is simply $(2N-j-1)t$. Thus, if we are standing on floor k $(1<k<N)$ in the one-elevator case under our simplifying assumptions, the expected waiting time, T_1, till the next elevator is given by the equation



$$T_1 = \sum_{j=1}^{2N-2} \frac{2N-j-1}{2N-2} t. \tag{3}$$

By symmetry, this may be written as

$$T_1 = t \sum_{j=1}^{2N-2} \frac{j}{2N-2}. \tag{4}$$

Using the well-known identity $\Sigma_{i=1}^{n} i = n(n+1)/2$, the equation in (4) can be simplified to

$$T_1 = \frac{(2N-1)(2N-2)t}{2(2N-2)} = \frac{(2N-1)t}{2}. \tag{5}$$

It is clear from equation (5) for T_1 that no matter what floor we are on (other than the top and bottom floors), the waiting time to an up elevator is constant. It is also apparent that (except at the top and bottom floors) the waiting time for a down elevator is the same as for an up elevator. Nonetheless, as we demonstrated above, the probability that the *next* elevator to come by will be going up depends on what floor we happen to be on (and how many floors are in the building), as specified in expressions (1) and (2). (If we assume you will always catch an elevator when it is waiting on your floor, the expression becomes

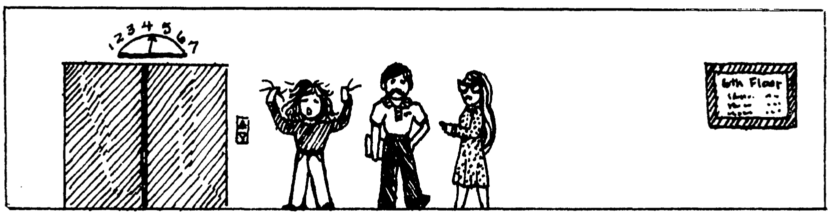
$$T_1 = \frac{2N-3}{2} t.$$

We can, of course, modify the above expression to take into account some probability, say one-half, of missing an elevator which is going in the desired direction waiting on your floor. We shall, however, neglect such complications.)

Now let us look at Puzzle 2. An analysis similar to that given above can shed light on the seeming paradox. Imagine that trains run every hour. The uptown comes at ten minutes to the hour, the downtown on the hour exactly. It's easy to see that our Lothario is five times more likely to go uptown than downtown—since only if he arrives in the ten minutes between when the uptown has just left and the downtown has not yet arrived will he end up going downtown. Of course, the expected waiting time will be one-half hour for both the uptown and downtown trains.

The solution to Puzzle 3 is essentially identical to that for Puzzle 2. Assume trains from each terminus depart on a fixed schedule (say, one every twelve hours). Our midwestern train buff is located at a distance from the western rail terminus (L.A.) and the eastern rail terminus (Chicago) such that the first train he sees is far more likely to be an eastbound train than a westbound train.

Now that the puzzles have been explained, let us continue our exploration of the behavior of elevators and of the people who wait for them. (This discussion may serve as an inquiry into a phenomenon which we term "elevator madness," by those who wait (and wait...) for an elevator.)



Let the conditional probability that a person who wants an elevator wants to go up given that he is on the kth floor be denoted by $p_k(U)$, and similarly let $p_k(D)$ equal the probability that someone on the kth floor who wants an elevator wants to go down. Let us initially assume that, except at rush hour, the relative attractiveness of all floors of the building is equal, i.e.,

$$p_k(U) = \frac{N-k}{N-1}$$

and

$$p_k(D) = \frac{k-1}{N-1}.$$

Such an assumption is not unreasonable for a single-company-owned office building.

We may define a frustration index f_k for someone on the kth floor waiting for an elevator as the likelihood that the next elevator is going in the wrong direction. For $0 < k < N$, we have

$$f_k = p_k(U)p(k\downarrow) + p_k(D)p(k\uparrow) \tag{6}$$

$$= \left(\frac{N-k}{N-1}\right)\left(\frac{N-k}{N-1}\right) + \left(\frac{k-1}{N-1}\right)\left(\frac{k-1}{N-1}\right), \ 1 < k < N.$$

After some straightforward algebra, we may rewrite f_k in the form

$$f_k = 1 - \frac{2(N-k)(k-1)}{(N-1)^2}. \tag{7}$$

It is clear from (7) that

$$f_k = f_{N-k+1}.$$

Since $(N-k)(k-1) > (N-k-1)k$ for $k > N/2$, it is easy to see from equation (7) that for N odd, f_k is minimal for $k = (N+1)/2$, $1 < k < N$. *Indeed, the closer you are to either the top or bottom floor of a one-elevator building (as long as you're not actually on the top or bottom floor), the more you can be expected to suffer from elevator madness.* Furthermore, from (7) it is straightforward to show that $f_{(N+1)/2} = 1/2$.

At the top or bottom floors of a building, the next elevator is always going in the desired direction, hence we may let $f_1 = f_N = 0$.

Under the above assumptions, average expected frustration, for the one-elevator case, which we shall denote by F_1, is given by

$$F_1 = \sum_{k=1}^{N} f_k \cdot p(k), \tag{8}$$

where $p(k)$ is the proportion of elevator seekers on the kth floor. If we assume that $p(k) = 1/N$ and continue assuming that all floors of the building are equally attractive, then using (7) and standard summation formulas, we may write F_1 as

$$F_1 = \frac{1}{N}\sum_{k=2}^{N-1} \frac{1}{(N-1)^2}\left[2k^2 - 2k(N+1) + N^2 + 1\right]$$

$$= \frac{(N-2)(2N-3)}{3N(N-1)}.$$

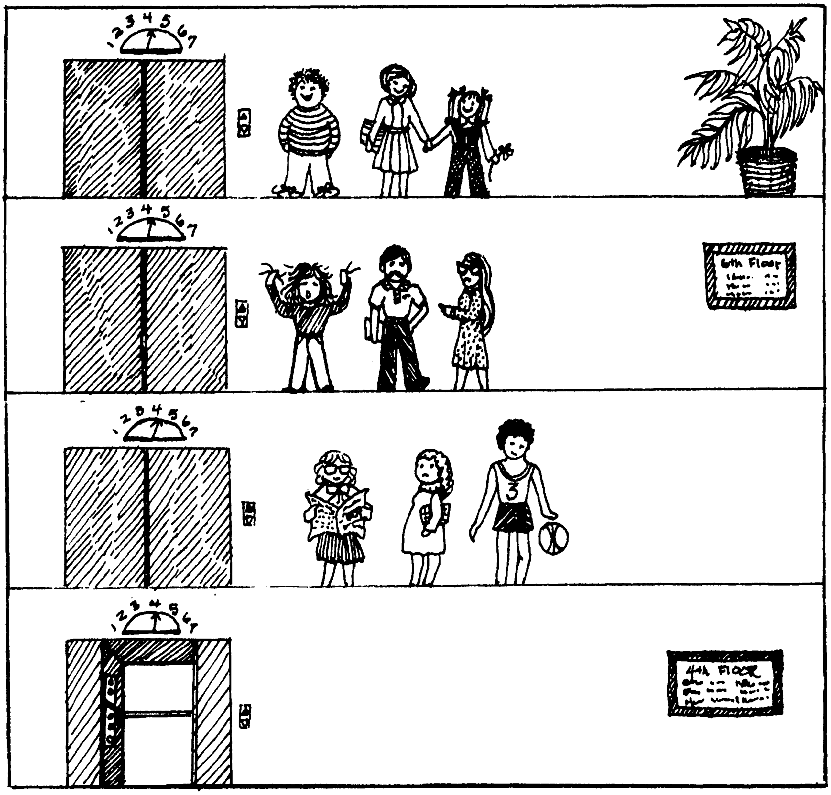
Although for small N ($N < 10$), F_1 is less than $1/2$, clearly as N gets larger, F_1 approaches $2/3$.

Let us now consider (still for the case $r = 1$) what happens during rush hours. During morning rush hour, every person is going up and during afternoon rush hours every person is going down. At morning rush hour, everyone begins on floor one, hence frustration (as we are measuring it) is zero. Afternoon rush hours are another story. At afternoon rush hour, if we assume as before that people are evenly spread through the building (i.e., $p(k) = 1/N$), we have, for $k < N$,

$$f_k = \frac{k-1}{N-1} \tag{9}$$

and

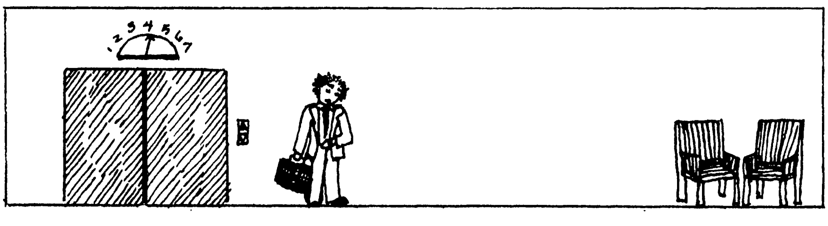
$$F_1 = \sum_{k=1}^{N-1} \frac{k-1}{N(N-1)} = \frac{(N-1)(N-2)}{2N(N-1)} = \frac{(N-2)}{2N}. \tag{10}$$



Some simple algebra suffices to show that if $k>(N+1)/2N$, then the "normal" frustration of waiting for an elevator (given by expression (6)) is less than the frustration in rush hour (given by expression (9)), while the reverse is true for $k<(N+1)/2N$. *Thus* (*in the single-elevator case*) *on the upper floors of a building, elevator madness should reach its peak in the late afternoon*. (We neglect complicating factors of tiredness, anxiousness to go home, etc., which may also be assumed to be maximal in the later afternoon. To the extent such factors exist, they merely strengthen our conclusion.) However, while F_1 approaches 2/3 for "normal" traffic, it is apparent from expression (10) that F_1 approaches 1/2 for rush-hour traffic. (We might also note that $\int_0^1 y\,dy = 1/2$.) Hence, on average, rush-hour traffic (under our simplifying assumptions) is less frustrating (in terms of the next elevator going the wrong way) than is "normal" traffic.

Most large buildings have more than one elevator, so that we should investigate what happens when there are $r \geqslant 2$ elevators (which serve all floors).

For $r=2$, for an individual on some specified floor k, let $q_1{}^{(k)}=$ the number of floors away elevator one is, in terms of its next appearance as a down [up] elevator; and define $q_2{}^{(k)}$ similarly.



It is clear that the number of floors away the *next* down [up] elevator is, is simply $\min(q_1^{(k)}, q_2^{(k)})$.

It is apparent that the probability that $\min(q_1^{(k)}, q_2^{(k)})$ equals h is independent of k, and thus we drop the superscript and assert that

$$\begin{aligned} p(\min(q_1,q_2)=h) &= p(q_1=h|q_2>h)p(q_2>h) \\ &\quad + p(q_1>h|q_2=h)p(q_2=h) \\ &\quad + p(q_1=h|q_2=h)p(q_2=h). \end{aligned}$$

This expression above may be rewritten as

$$\begin{aligned} p(\min(q_1,q_2)=h) &= \frac{1}{2(N-1)}\left[\frac{(2N-2-2h)}{2(N-1)}+\frac{(2N-2-h)}{2(N-1)}+\frac{1}{2(N-1)}\right] \\ &= \frac{4N-3-2h}{4(N-1)^2}. \end{aligned}$$

When it takes t minutes for the elevator to travel between consecutive floors (and waiting time is neglected), it is clear that T_2, the expected waiting time for the next down [up] elevator in the case where there are two elevators, is given by

$$T_2=\sum_{h=1}^{2N-2}\left(\frac{4N-3-2h}{4(N-1)^2}\right)ht=\frac{(2N-1)(4N-3)t}{12(N-1)}. \tag{11}$$

A much simpler way to get the same result is to recognize that

$$p(\min(q_1,q_2)=h)=\frac{h}{2N-2}$$

and hence

$$T_2=\sum_{h=1}^{2N-2}\frac{h^2}{4(N-1)^2}=\frac{(2N-1)(4N-3)t}{12(N-1)}.$$

More generally, when there are r elevators,

$$p(\min(q_1,q_2,\ldots,q_r)=h)=\sum_{h=1}^{2N-2}h^r$$

and hence

$$T_r=\sum_{h=1}^{2N-2}\frac{h^r}{[2(N-1)]^r}.$$

While there are several formulae to obtain Σh^r (see [3]) it is much simpler to assume N large and obtain the very useful result that

$$T_r\approx Nt\int_0^2\left(\frac{x^r}{2}\right)dx=\frac{2Nt}{r+1}. \tag{12}$$

Note that for $r=1$, the approximation in (12) gives $T_1\approx Nt$, which compares favorably with our discrete value expression, $T_1=(N-\frac{1}{2})t$. For $r=2$, (12) gives $T_2\approx(2Nt)/3$, the same result obtained by taking the limit of expression (11) as $N\to\infty$. As we would expect, the more elevators there are in operation, the less is the expected waiting time to the next up [down] elevator.

For $r=2$, we may obtain expressions $p_2(k\uparrow)$ and $p_2(k\downarrow)$ for the probability that the next elevator to come to floor k will be an up or a down elevator, respectively. Because there are two elevators, we also now have the probability of a tie, i.e., the up elevator and the down elevator arriving simultaneously, which we shall denote $p_2(k{\updownarrow})$. For $r=2$, we may set up the problem in terms of a decision tree. (See FIGURE 1.)

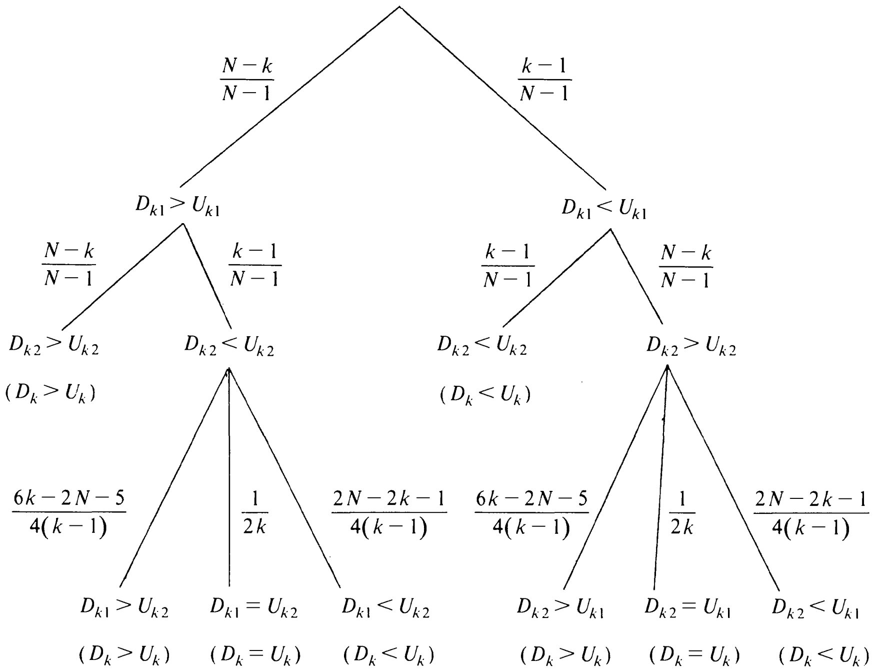


FIGURE 1. **Probability that the next elevator will be going up/down, if you're on Floor k and there are two elevators in the building ("$>$" denotes "arrives before"). Probabilities shown in bottom portion of FIGURE 1 are for $k \geqslant (N+1)/2$.**

In FIGURE 1, D_{ki} (respectively U_{ki}) denotes the ith elevator arrives on the kth floor, going down (respectively, up), and we use the symbol $>$ to denote "arrives before." $D_{ki} > U_{ki}$ denotes the event: when elevator i reaches floor k, it is going down rather than up. $D_{ki} < U_{ki}$ and $D_{ki} = U_{kj}$ are similarly defined. The outcomes are shown on the terminal nodes of the decision tree. For example, if $D_{k1} > U_{k1}$ and $D_{k2} > U_{k2}$, then the next elevator to reach floor k must be a down elevator, an outcome which we have denoted $D_k > U_k$. The probabilities shown in this tree can be straightforwardly derived. The only interesting cases are those where one elevator reaches the kth floor first coming down while the other elevator reaches it first going up. We need to calculate various conditional probabilities, e.g., $p(D_{k2} > U_{k1} | D_{k1} < U_{k1} \wedge D_{k2} > U_{k2})$. If $D_{ki} < U_{ki}$, then the maximum distance (in floors) to the arrival of elevator i on floor k as a down elevator is $2(N-k)$. If that were not so then elevator i would have come to floor k first as a down elevator. (Hint: try to visualize why this must be so.) Analogously, if $D_{ki} > U_{ki}$, then the maximum distance (in floors) to the arrival of elevator i on floor k as an up elevator is $2(k-1)$. Thus for $k \geqslant (N+1)/2$

$$\begin{aligned} p(D_{k2} > U_{k1} | D_{k1} < U_{k1} \wedge D_{k2} > U_{k2}) &= \sum_{h=1}^{2(N-k)} p(q_2 > h | q_1 = h) p(q_1 = h) \\ &= \sum_{h=1}^{2(N-k)} \left[\left(\frac{2(k-1)-h}{2(k-1)} \right) \left(\frac{1}{2(N-k)} \right) \right] \\ &= \frac{6k-2N-5}{4(k-1)}. \end{aligned}$$

The other probability values specified in FIGURE 1 are calculated in a similar fashion. Performing the necessary algebra, we eventually obtain, for $k \leqslant (N+1)/2$

$$p_2(k\downarrow)=\frac{(N-k)(4k-5)}{2(N-1)^2} \tag{13}$$

$$p_2(k\uparrow)=\frac{2(k-1)^2+(N-k)(2N-2k-1)}{2(N-1)^2}. \tag{14}$$

The probability of a tie is given by

$$p_2\left(k_{\downarrow}^{\uparrow}\right)=\frac{N-k}{(N-1)^2}.$$

For $k>(N+1)/2$, we reverse equations (13) and (14). Of course for $k=(N+1)/2$ those two equations give identical values.

Once again, we may define a frustration index f_k:

$$\begin{aligned} f_k &= p_k(U)p_2(k\downarrow)+p_k(D)p_2(k\uparrow) \\ &= \frac{(N-k)^2(4k-5)+(k-1)\left[2(k-1)^2+(N-k)(2N-2k-1)\right]}{2(N-1)^3}. \end{aligned}$$

Comparing equations (13) and (14) with equations (1) and (2) we observe that for $k<(N+1)/2$, $p_2(k\uparrow)<p(k\uparrow)$, while $p_2(k\downarrow)>p(k\downarrow)$; while the reverse is true for $k>(N+1)/2$. What this means is that the addition of a second elevator has only reduced the likelihood that the next elevator is up [down] for half of the floors (the lower floors). *Additional elevators tend to "even out" the probabilities that the next elevator to reach a floor will be going up as opposed to going down.*

For r and N large, useful approximations are

$$p(k\uparrow)=p(k\downarrow)\approx\frac{(2N-3)}{4(N-1)}\approx\frac{1}{2}$$

$$p\left(k_{\downarrow}^{\uparrow}\right)=\frac{1}{2(N-1)}.$$

Hence, for r and N large

$$f_k\approx\frac{1}{2}$$

$$F_r\approx\frac{1}{2}.$$

Elevators, trains, and subways are closed-loop systems. The puzzles we have discussed in this paper illuminate the difference between "frequency" and "phase" [**1**]. The reader might try to think of other examples of closed-loop systems and the mathematical problems they pose. For example, subways in New York City have both express trains and local trains (the former are faster because they make fewer stops), and you might try to formulate conditions in which it makes sense to take a train going in the wrong direction. (Hint: do you have to be at a station at which only local trains stop?)

I wish to acknowledge a considerable debt of gratitude to an anonymous referee, who provided a clear formulation of the solution to the elevator problem when the number of floors is large and who corrected a number of errors in the original statement of the equations in this paper, and to my colleague Louis Narens, who has convinced me that madness and mathematics can happily coexist.

References

[1] George Gamow and Marvin Stern, Puzzle-Math, Viking, New York, 1958.

[2] Frederick Mosteller, Fifty Challenging Problems in Probability, with Solutions, Addison-Wesley, Reading, MA, 1965.

[3] J. W. Paul, On the sum of the products of the first integers, Amer. Math. Monthly, 78 (1971) 271–272.

Permutable Primes

JAN LIST BOAL
JEAN H. BEVIS
Georgia State University
Atlanta, GA 30303

Prime numbers and their properties have fascinated both novice and professional mathematicians in every age. Of particular interest in recent years are those primes whose digits can be rearranged to produce other primes. Such primes include palindromic primes [**5**] which are left unchanged by reversing their digits, and absolute primes [**3**], [**6**] which remain prime under all permutations of their digits. Using only two types of rearrangements, reversal of digits and cyclic permutations, we introduce the classes of reversible, cyclic, and symmetric primes. First we show the relationships between these five classes of permutable primes, and note the importance of cyclic primes to the study of absolute primes. Then, based on data from experiments, we propose an asymptotic formula for the number of reversible primes with n digits. If this formula is correct, it could lead to a proof of the conjecture of Gabai and Coogan [**4**] that there are infinitely many palindromic primes.

An **absolute prime** (base ten) is any prime such that every permutation of its digits produces a prime. Obviously each of the primes 2, 3, 5, 7, 11 is an absolute prime. We list in TABLE 1 all known absolute primes having two distinct digits, with primes having the same digits grouped in families.

$n=2$:	$\{13,31\}$, $\{17,71\}$, $\{37,73\}$, $\{79,97\}$
$n=3$:	$\{113,131,311\}$, $\{199,919,991\}$, $\{337,373,733\}$.

TABLE 1. **Absolute primes with n digits**

It has been shown by Johnson [**6**] that if there is an absolute prime with two distinct digits that is not in TABLE 1, it will have more than nine billion digits. The only other absolute primes are the prime repunits. **Repunits** [**1**], [**2**], [**5**], [**8**], [**9**] are numbers all of whose digits are ones and are denoted by R_n where n is the number of digits. Besides $R_2=11$, it is known that R_{19}, R_{23}, and R_{317} are prime (see Yates [**9**]). The primality of the last of these, R_{317}, was discovered only recently and came as a surprise [**7**].

A slightly less restrictive condition on permutation of digits defines the set of cyclic primes. A **cyclic prime** is a prime such that every cyclic permutation of its digits is also prime. All absolute primes are cyclic primes. Using an Algol W program on a UNIVAC 90/80 computer we found the families of cyclic primes which are not absolute, shown in TABLE 2. Primality was tested using a list of primes generated by a subroutine based on a modified sieve of Eratosthenes. Using this program we also found that there are no cyclic primes with seven or eight digits.

$n=3$:	$\{197, 719, 971\}$
$n=4$:	$\{1193, 1931, 9311, 3119\}$, $\{3779, 7793, 7937, 9377\}$
$n=5$:	$\{11939, 19391, 93911, 39119, 91193\}$,
	$\{19937, 99371, 93719, 37199, 71993\}$
$n=6$:	$\{193939, 939391, 393919, 939193, 391939, 919393\}$,
	$\{199933, 999331, 993319, 933199, 331999, 319993\}$.

TABLE 2. **Cyclic primes with n digits that are not absolute primes**

An absolute or cyclic prime of two or more digits may only contain the digits 1, 3, 7, 9. Otherwise some permutation of the number ends in 0, 2, 4, 5, 6, or 8, and is divisible by 2 or 5. Bhargava and Doyle [**3**] have shown that no absolute prime may contain all four of the digits 1, 3, 7, 9. Johnson [**6**] has extended this result and proved (i) no absolute prime may contain three of the four digits 1, 3, 7, 9, and (ii) no absolute prime may contain two or more of each of two digits from 1, 3, 7, 9. Several cyclic primes in TABLE 2 (e.g., 11939) show that neither of Johnson's results holds for cyclic primes. The cyclic prime 19937 shows that a cyclic prime may contain all four of the digits 1, 3, 7, 9. The prime 19937 is also the twenty-fourth Mersenne exponent and is the largest known Mersenne exponent which is also cyclic.

A **symmetric** or **dihedral prime** is a prime such that every symmetric permutation of its digits also yields a prime. To obtain symmetric permutations of an n-digit number, imagine the digits attached in cyclic order to the vertices of an n-sided regular polygon and permute the digits according to the rotations and reflections that leave the polygon fixed. Rotations correspond to cyclic permutations of the digits. The reflection about the perpendicular bisector of the side of the polygon which connects the vertices corresponding to the first and last digits of the number reverses the digits of the number (see FIGURE 1). The symmetric permutations form the dihedral subgroup of the group of all permutations and are generated by the rotations and any single reflection. Thus *a number is a symmetric prime if and only if the number and its reverse are both cyclic primes*. The numbers 11939 and 193939 generate two families of symmetric primes that are not absolute primes. The only other known symmetric primes are the absolute primes.

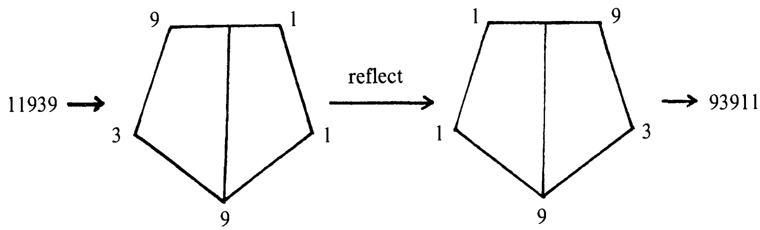


FIGURE 1

A **palindrome** is a string of characters which is the same when read from right to left as when read from left to right. A **palindromic prime** is a prime with this property. **Reversible primes** are primes which are also prime when read from right to left. Palindromic, symmetric, and absolute primes are all special classes of reversible primes. The relationships between the classes of permutable primes is illustrated in FIGURE 2. We note that the intersection of the class of reversible primes with the class of cyclic primes is larger than the class of symmetric primes since 37199 is cyclic and reversible, but its reverse is not cyclic. However, the intersection of the class of palindromic primes with the class of cyclic primes is contained in the class of symmetric primes. This is clear since the reverse of such a number is itself.

One relationship illustrated in FIGURE 2 merits special attention. From Johnson's results [**6**], every absolute prime must be a repunit (called **type A** by Johnson) or a number of the form $aR_n+(b-a)10^m$ (called **type B**) where a and b are distinct digits selected from 1, 3, 7, 9 and $0\leqslant m\leqslant n$. Thus a type B number consists of $n-1$ digits a and one digit b. All distinct permutations of a type B number may be obtained as cyclic permutations. Hence,

THEOREM. *If x is a type* A *or a type* B *number, then the following are equivalent*:

(i) *x is a cyclic prime*,

(ii) *x is a symmetric prime*,

(iii) *x is an absolute prime*.

The results of our computer search suggest that the classes of absolute, symmetric, and cyclic

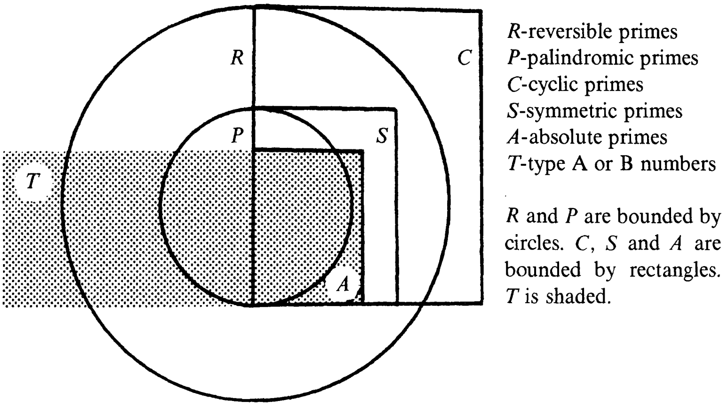


FIGURE 2

primes may all be finite. Establishing this result is likely to be a difficult problem. An easier problem would be to prove that there are only a finite number of twin primes one or both of which are cyclic or absolute. The pair $\{197, 199\}$ is the largest pair of twin primes each of which is known to be cyclic.

Gabai and Coogan [**4**] suggest that there may be infinitely many palindromic primes, and that there may be a larger relative percentage of primes in the set of palindromic numbers than in the set of positive integers. This behavior could be a special case of a more general property, namely, that the reverse of a number is more likely to be a prime if the original number is prime. To test this conjecture, consider the following experiments.

EXPERIMENT 1: Choose at random a number with n digits that does not end in the digit 0 (so that the reverse also has n digits), and reverse the digits. The probability of obtaining a prime is $p(n)/(81 \times 10^{n-2})$ where $p(n)$ is the number of primes with n digits. One can increase the chances of obtaining a prime by modifying Experiment 1.

EXPERIMENT 2: Choose a prime number with n digits and reverse the digits. The probability of obtaining a prime is $r(n)/p(n)$ where $r(n)$ is the number of reversible primes with n digits. By a computer search, we obtained values of $r(n)$ for $n \leqslant 7$. By comparing columns 4 and 5 of TABLE 3 we see that for $n \leqslant 7$ the reverse of a number is more likely to be prime if the original number is prime. This behavior may be partially explained by a property of multiples of 3. A number is a multiple of 3 if and only if the sum of its digits is a multiple of 3. Thus a number is divisible by 3 if and only if its reverse is divisible by 3. It is also known that a number is divisible by 11 if and only if its reverse is divisible by 11. Experiment 2 eliminates any multiple of 3 or 11, and hence the reverse is not divisible by 3 or 11 and is more likely to be prime. In order to test whether or not this completely explains the observed behavior of $r(n)$ we modify Experiment 1 again.

EXPERIMENT 3: Choose a number with n digits that is not divisible by 3, 10, or 11 and reverse its digits. Of the possible choices of numbers in Experiment 1, only $(20/33)(81 \times 10^{n-2})$ of these are possible choices in this experiment. Thus, the probability of obtaining a prime in Experiment 3 is $p(n)/(54/11 \times 10^{n-1})$. By comparing columns 5 and 6 of TABLE 3 we see that (except for $n = 2$) column 5 has larger values when n is odd, and that column 6 has larger values when n is even. This variation is most likely due to the fact that, except for 11, there are no palindromic primes with an even number of digits [**4**].

The results of our computations suggest an asymptotic formula for $r(n)$, namely

$$r(n) \approx 11p^2(n)/(54 \times 10^{n-1}).$$

n	$p(n)$	$r(n)$	$\frac{p(n)}{(81\times 10^{n-2})}$	$\frac{r(n)}{p(n)}$	$\frac{p(n)}{(54/11\times 10^{n-1})}$
2	21	9	.259	.429	.428
3	143	43	.177	.301	.291
4	1061	204	.131	.192	.216
5	8363	1499	.103	.179	.170
6	68906	9538	.085	.138	.141
7	586081	71142	.074	.121	.119

TABLE 3

If this formula is correct, it would prove the existence of an infinite number of reversible primes, and might lead to the same result for palindromic primes.

The authors would like to thank the editor and the referees for their helpful suggestions which did much to improve the format of this paper.

References

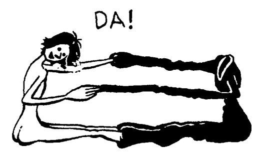
[1] Albert H. Beiler, Recreations in the Theory of Numbers, Dover, New York, 1964.
[2] John Brillhart and J. L. Selfridge, Some factorizations of $2^n \pm 1$ and related results, Math. Comp., 21 (1967) 87–96.
[3] T. N. Bhargava and P. H. Doyle, On the existence of absolute primes, this MAGAZINE, 47 (1974) 233–234.
[4] H. Gabai and D. Coogan, On palindromes and palindromic primes, this MAGAZINE, 42 (1969) 252–254.
[5] Martin Gardner, Mathematical games, Sci. Amer., 210 (March 1964) 127; and *ibid.*, 210 (June 1964) 118.
[6] A. W. Johnson, Problem 955, this MAGAZINE, 50 (1977) 100–103.
[7] H. C. Williams, Some primes with interesting digit patterns, Math. Comp., 32 (1978) 1306–1310.
[8] Samuel Yates, Peculiar properties of repunits, J. Rec. Math., 2 (1968–1969) 139–146.
[9] ______, Mystique of repunits, this MAGAZINE, 50 (1978) 22–28.

∀ and ∃

Said an upside-down A to an inside-out E,
"*Universal*'s the epithet measuring me.
Your scope is so small
Compared with *For all*—
There is is no more than a form of *To be*."

Said the inside-out E to the upside-down A,
"*To be* is the question! Avoid it? No way!
My role's existential—
I'm basic, essential—
Of equal importance the parts that we play!"

Said the upside-down A *and* the inside-out E,
"Let's drop the dispute and agree to agree!
A nifty notation
Reversed in negation—
Our tandem performance is something to see!"



—KATHARINE O'BRIEN

n	$p(n)$	$r(n)$	$\frac{p(n)}{(81\times 10^{n-2})}$	$\frac{r(n)}{p(n)}$	$\frac{p(n)}{(54/11\times 10^{n-1})}$
2	21	9	.259	.429	.428
3	143	43	.177	.301	.291
4	1061	204	.131	.192	.216
5	8363	1499	.103	.179	.170
6	68906	9538	.085	.138	.141
7	586081	71142	.074	.121	.119

TABLE 3

If this formula is correct, it would prove the existence of an infinite number of reversible primes, and might lead to the same result for palindromic primes.

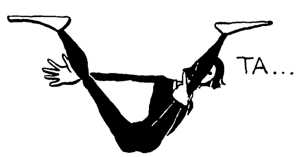
The authors would like to thank the editor and the referees for their helpful suggestions which did much to improve the format of this paper.

References

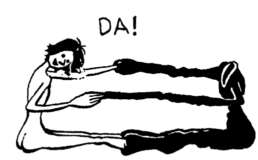
[1] Albert H. Beiler, Recreations in the Theory of Numbers, Dover, New York, 1964.
[2] John Brillhart and J. L. Selfridge, Some factorizations of $2^n \pm 1$ and related results, Math. Comp., 21 (1967) 87–96.
[3] T. N. Bhargava and P. H. Doyle, On the existence of absolute primes, this MAGAZINE, 47 (1974) 233–234.
[4] H. Gabai and D. Coogan, On palindromes and palindromic primes, this MAGAZINE, 42 (1969) 252–254.
[5] Martin Gardner, Mathematical games, Sci. Amer., 210 (March 1964) 127; and *ibid.*, 210 (June 1964) 118.
[6] A. W. Johnson, Problem 955, this MAGAZINE, 50 (1977) 100–103.
[7] H. C. Williams, Some primes with interesting digit patterns, Math. Comp., 32 (1978) 1306–1310.
[8] Samuel Yates, Peculiar properties of repunits, J. Rec. Math., 2 (1968–1969) 139–146.
[9] ______, Mystique of repunits, this MAGAZINE, 50 (1978) 22–28.

∀ and ∃

Said an upside-down A to an inside-out E,
"*Universal*'s the epithet measuring me.
Your scope is so small
Compared with *For all*—
There is is no more than a form of *To be*."



Said the inside-out E to the upside-down A,
"*To be* is the question! Avoid it? No way!
My role's existential—
I'm basic, essential—
Of equal importance the parts that we play!"



Said the upside-down A *and* the inside-out E,
"Let's drop the dispute and agree to agree!
A nifty notation
Reversed in negation—
Our tandem performance is something to see!"

—KATHARINE O'BRIEN

How to Win (or Cheat) in the Solitaire Game of "Clock"

MICHAEL W. ECKER
Pennsylvania State University
Worthington Scranton Campus
Dunmore, PA 18512

As a child, I learned the solitaire game of "Clock" and found that I lost this game of pure luck most of the time. It was not until recently, two decades later, that I discovered why.

Since minor variations in the game exist, note that the "Clock" game I play goes as follows: a shuffled deck of cards is dealt out face down to form 13 piles of 4 cards each, with each pile assigned a number. The top card of pile #1 is turned over. After the face value of the card is noted, it is discarded. Then one proceeds to the pile having the same number as the face of the card just discarded (e.g., if a jack was turned up, proceed to pile #11). Now repeat. The game ends when you are directed to an empty pile. The object of the game is to win by having no cards left when this happens.

Recently problems have been posed and answered concerning "Clock"; all have been probabilistic in nature [**1**], [**2**], [**3**]. My frustrating experience in playing the game led me to ask instead a game-theory question: can you predict in advance whether a given distribution of cards will result in a win without actually playing? We permit ourselves the luxury of looking at some subset of the distributed cards in order to make this prediction. (Otherwise we are dealing with so-called psychic phenomena, not mathematics!) The answer to the question is yes; in fact, we will show that *one can determine whether one has a win by an inspection of the bottom card of each pile.*

As in [**2**] and [**3**], it is no more difficult to consider the slightly more general game of "Clock" where there are L piles of cards having k cards each. It is understood that there are k copies of each of the cards having values $1,2,\dots,L$. The key to analyzing generalized "Clock" is to understand that the game always ends with the player being directed to the empty pile #1, regardless of whether the game is a loss or a win. The game cannot end at a different pile, for this reason: after a pile of cards, say the jth, for some $j=2,3,\dots,L$, becomes empty, we must have used all k of the cards with face value j in order to have been directed to that pile k times. Hence, there are no cards numbered j left, so there is no way that we can ever be directed back to this now empty pile. Note how pile #1 is distinguished in that the first card of it is turned up in the beginning of the game without having had a card directing us to this pile, which explains why the foregoing argument does not apply to $j=1$.

We now show how the bottom cards of each pile determine the outcome of the game. Assume the cards are distributed into L piles. Define a function f which maps the set $\{1,2,\dots,L\}$ into itself as follows: f(pile number) = face value of the card at the bottom of that pile. Although in general f will not be a permutation, we can use f to define cycles as follows. An f-cycle of length n is an ordered n-tuple $(a_1,a_2,\dots,a_n)$, $1\leqslant n\leqslant L$, with $f(a_1)=a_2$, $f(a_2)=a_3,\dots,f(a_{n-1})=a_n$, and $f(a_n)=a_1$. We will show:

THEOREM. *A distribution of cards is a winning one if and only if every f-cycle contains the number* 1. (*Thus if an f-cycle can be found not containing* 1, *the distribution is a losing one.*)

Some examples, each with $k=4$, $L=5$, will illustrate the Theorem.

EXAMPLE 1. Bottom cards, in respective order of piles: 2,3,4,2,1. Here we have $f(2)=3$, $f(3)=4$, $f(4)=2$, so (2,3,4) is an f-cycle not containing 1. It is easy to visualize this using permutation notation: $\begin{pmatrix}2&3&4\\3&4&2\end{pmatrix}$. The Theorem says we lose this game.

EXAMPLE 2. Bottom cards: 2,1,1,3,4. The only f-cycles here are (1,2) and (2,1), so the Theorem says we win this game.

EXAMPLE 3. Bottom cards: 1,2,1,5,3. Since $f(2)=2$, the cycle of length one, (2), is an f-cycle not containing 1, so we lose this game.

In Example 3, f has a fixed point $j \neq 1$; this is a losing game, regardless of the other cards. Before producing a proof of the Theorem, we should see why the bottom cards dictate the fate of the game. We can illustrate the ideas using Example 3, the extreme fixed-point case of $f(j)=j$, for some $j \neq 1$.

In Example 3, we lose because the "2" at the bottom of pile #2 can never be drawn. For, each time we draw a "2", we proceed to pile #2. Thus, in order to draw the fourth (and final) card from pile #2, we would need a fourth "2" to direct us to this pile. But this is impossible, since one of the four "2"s sits at the bottom of this very pile!

This observation can now be readily generalized. Suppose first that a given distribution produces an f-cycle not containing 1; i.e., we have cards $a_1, a_2, \ldots, a_n$ at the bottom of piles $a_n, a_1, a_2, \ldots, a_{n-1}$, respectively. We contend that no bottom card a_i in this cycle will ever be drawn. For if some a_i were drawn first of all these, since it is at the bottom of pile $a_{i-1}(\text{mod } n)$ we would have drawn the last card a_{i-1} to be directed to pile $\#a_{i-1}$, a contradiction to assumption about a_i being drawn first. Note that this argument includes the case of a cycle of length $n=1$ (illustrated by Example 3). Hence, the presence of an f-cycle not containing 1 indicates a losing game.

Now, for the converse, suppose that a distribution is a losing one. Then there will be at least one nonempty pile left, and not #1. (Recall the game ends when the player is directed to an empty pile #1.) Let's enumerate these for definiteness as pile numbers $a_1, a_2, \ldots, a_m$. Regardless of how many cards there are left in any of these piles, the remaining cards at the bottom of the piles must bear values which form a subset of these a_i's. For, if a card with face value b is at the bottom of a pile, then pile b could never have been exhausted. To show that an f-cycle must exist, consider the card at the bottom of pile $\#a_1$. If it is numbered a_1, we are done. Otherwise, it is some a_j. Look at pile $\#a_j$. If the bottom card is a_1 or a_j, we are done. Otherwise, it is some distinct a_k. Continuing the process, the pigeon-hole principle insures that we must eventually get a repeated value or a cycle of length m. In either case, we have our f-cycle, and it does not contain 1, since $a_i \neq 1$, $1 \leq i \leq m$. This completes the proof of the characterization Theorem.

As an interesting aside, it should be clear that any distribution not having a "1" at the bottom of some pile other than #1, must be a loser. (Repeat or mimic the above pigeon-hole argument to verify this.) However, this situation implies the existence of an f-cycle not containing 1 and hence this necessary but *not* sufficient condition for a win is completely subsumed under the condition stated in the Theorem.

In closing, it is worth mentioning that the foregoing characterization lends itself to a probabilistic argument to prove that the probability of a win in generalized "Clock" is $1/L$ ($1/13$ for ordinary "Clock"). However, such a proof is tedious, and so I refer the reader to the crisper proof in [**2**].

References

[1] T. A. Jenkyns and E. R. Muller, A probabilistic analysis of Clock Solitaire, this MAGAZINE, 54 (1981) 202–208.

[2] David Kleiner, Solution to Problem No. 1066, this MAGAZINE, 53 (1980) 184–185.

[3] Eric Mendelsohn and Stephen Tanny, Problem No.1066, this MAGAZINE, 52 (1979) 113.

PROBLEMS

DAN EUSTICE, Editor
LEROY F. MEYERS, Associate Editor
The Ohio State University

Proposals

To be considered for publication, solutions should be mailed before June 15, 1982.

1136. Let $\mathbb{C}^*$ be the multiplicative group of the nonzero complex numbers and let S^1 be the subgroup of $\mathbb{C}^*$ where the elements of S^1 have modulus one. Are $\mathbb{C}^*$ and S^1 isomorphic as abstract groups? [*William Horten and Daniel B. Shapiro, The Ohio State University.*]

1137. It is known that $\tan x + \sin x \geqslant 2x$ for $0 \leqslant x < \pi/2$, which is a stronger inequality than $\tan x \geqslant x$. Establish the still stronger inequality

$$a^2 \tan x (\cos x)^{1/3} + b^2 \sin x \geqslant 2xab$$

for $0 \leqslant x \leqslant \pi/2$. [*M. S. Klamkin, University of Alberta.*]

1138. Let X be a nonsingular matrix with columns $X_1, X_2, \ldots, X_n$. Let $Y = [X_2, X_3, \ldots, X_n, \mathbf{0}]$. Show that the matrices $A = YX^{-1}$ and $B = X^{-1}Y$ have rank $n-1$ and have only 0 for eigenvalues. Conversely, show that every $n \times n$ matrix A of rank $n-1$ and with only 0 for eigenvalues can be written $A = YV$ for some nonsingular $X = V^{-1}$ and Y defined as above. [*John Z. Hearon, National Institutes of Health.*]

1139. Let $p(x)$ be a polynomial with rational coefficients and suppose $p(x)$ is irreducible over the rationals. Let α be a complex number such that $p(\alpha) = 0$. Then we know $p(x) = (x - \alpha)q(x)$, where $q(x)$ is a polynomial with complex coefficients. The leading coefficient of $q(x)$ is rational. Can any of the other coefficients of $q(x)$ be rational numbers? [*Roger L. Creech, East Carolina University.*]

ASSISTANT EDITORS: DON BONAR, *Denison University;* WILLIAM A. MCWORTER, JR., *The Ohio State University.* *We invite readers to submit problems believed to be new. Proposals should be accompanied by solutions, when available, and by any information that will assist the editors. Solutions to published problems should be submitted on separate, signed sheets. An asterisk (*) will be placed by a problem to indicate that the proposer did not supply a solution. A problem submitted as a Quickie should be one that has an unexpected succinct solution. Readers desiring acknowledgment of their communications should include a self-addressed stamped card. Send all communications to this department to Leroy F. Meyers, The Ohio State University, 231 W. 18th Ave., Columbus, Ohio 43210.*

Quickies

Solutions to Quickies appear at the conclusion of the Problems section.

Q669. Taking the formula for area in polar coordinates, $\int r^2(\theta)\,d\theta/2$ and the ellipse

$$x = 4\cos\theta,\ y = 3\sin\theta,\ 0 \leqslant \theta \leqslant 2\pi,$$

one finds $\int_0^{2\pi}(4^2\cos^2\theta + 3^2\sin^2\theta)\,d\theta/2$ is $25\pi/2 = 12.5\pi$ and not $\pi\cdot 4\cdot 3 = 12\pi$. What's wrong? [*Student in calculus, The Ohio State University.*]

Q670. Let N be the largest integer such that the decimal expansions of both N and $7N$ have exactly 100 digits each. Find the 50th digit from the left of N. [*Stanley Friedlander, Bronx Community College.*]

Solutions

Squared Sines **November 1980**

1107. Determine the maximum value of $\sin A_1 \sin A_2 \cdots \sin A_n$ if $\tan A_1 \tan A_2 \cdots \tan A_n = 1$. [*M. S. Klamkin, University of Alberta.*]

Solution: Let P denote the product of the sines. From $\Pi_1^n \tan A_i = 1$, we have

$$P = \prod_1^n \sin A_i = \prod_1^n \cos A_i.$$

Then

$$P^2 = \prod_1^n \sin A_i \cos A_i = 2^{-n}\prod_1^n \sin 2A_i \leqslant 2^{-n}.$$

Thus $|P| \leqslant 2^{-n/2}$ and equality is attained when $A_1 = A_2 = \cdots = A_n = \pi/4$. Therefore $2^{-n/2}$ is the maximum value.

Jeremy D. Primer
Columbia High School
Maplewood, New Jersey

Also solved by Mangho Ahuja, Anders Bager (Denmark), Gary Birkenmeier, Benny Cheng, Chico Problem Group, Chi Hoi Duong, Philip M. Dunson, Edilio Escalona (Venezuela), Nick Franceschine III, Jerrold W. Grossman, Steven Gustafson, Robert M. Hashway, G. A. Heuer & Karl Heuer, Victor Hernandez (Spain), Hans Kappus (Switzerland), L. Kuipers (Switzerland), Allen Kwiatkowski, Russell Lyons, Beatriz Margolis (France), Larry Olson, Jon Shreve, Michael Vowe (Switzerland), Michael Woltermann, Yan Loi Wong (Hong Kong), Ken Yocum, and the proposer.

Counting Cycles November 1980

1108. For n odd, let $C[n]$ be the number of cycles in the permutation of $\{0,1,\dots,n-1\}$ sending $i \to 2i \pmod n$. Prove that $C[3(2^j-1)] = C[5(2^j-1)]$ for all odd positive integers j. [*James Propp, Harvard University.*]

Solution: Let $n = 2^j - 1$. The assertion will follow from the more general result that if p is an odd prime and the order of $2 \pmod p$ is $p-1$ and $\gcd(p-1, j) = 1$, then $C[pn] = 2C[n]$.

Write the given permutation T as a product $(0)Z_1Z_2\cdots Z_r$ of disjoint cycles. If $Z_i = (a, b, c, \dots)$, let $pZ_i = (pa, pb, pc, \dots)$. Then pZ_i is a cycle of $T'\colon i \to 2i \pmod{pn}$.

Since the order of $2 \pmod p$ is $p-1$, and $\gcd(p-1, j) = 1$, $p \nmid n$, and so $0' = (n, 2n, \dots, 2^{p-2}n)$ is a cycle of T' different from any of the pZ_i.

Next, each $Z_i = (a, b, c, \dots)$ contains a number x which is not divisible by p. If this were not the case, and if s were the smallest positive integer such that $2^s a > n$, then $x = 2^s a - n$ would be in Z_i, but $p \nmid x$ since $p \nmid n$. Let Z_i' be the cycle of x in T'. The Z_i''s are mutually disjoint. Indeed, if $x_i \in Z_i, x_k \in Z_k, p \nmid x_i, x_k$, and $i \neq k$, then $2^s x_i \not\equiv x_k \pmod n$ for all integers s, and thus $2^t x_i \not\equiv x_k \pmod{pn}$ for all integers t. Since cycles are either disjoint or identical, and $x_i \in Z_i'$, but $x_i \notin Z_k'$, it follows that $Z_i' \cap Z_k' = \varnothing$ for $i \neq k$. The length of Z_i' is the smallest positive integer m such that $2^m x_i \equiv x_i \pmod{pn}$, i.e., the order of $2\left(\bmod \frac{pn}{(pn, x_i)}\right)$. Since p and n are relatively prime, the order of $2\left(\bmod\, p\frac{n}{(pn, x_i)}\right)$ is the least common multiple of the order of $2 \pmod p$ and the order of $2\left(\bmod \frac{n}{(pn, x_i)}\right)$. The order of $2 \pmod p$ is $p-1$, and the order of $2\left(\bmod \frac{n}{(pn, x_i)}\right) =$ the order of $2\left(\bmod \frac{n}{(n, x_i)}\right)$ is relatively prime to $p-1$ (it divides j, since $n = 2^j - 1$), and so the length of Z_i' is $p-1$ times the order of $2\left(\bmod \frac{n}{(n, x_i)}\right)$, i.e., the length of Z_i' is $p-1$ times the length of Z_i.

Thus the Z_i''s taken together account for $(p-1)(n-1) = pn - p - n + 1$ numbers from $\{0, 1, \dots, pn-1\}$. (0), $0'$ and the pZ_i''s account for the remaining $1 + (p-1) + (n-1) = p + n - 1$ numbers, and so

$$T' = (0)0'pZ_1Z_1'pZ_2Z_2'\cdots pZ_rZ_r'.$$

Thus $C[pn] = 2C[n]$.

MICHAEL WOLTERMANN
Washington & Jefferson College

Also solved by Daniel Shapiro. Shapiro proved the following generalization: Suppose n_1 and n_2 are odd positive integers with $\gcd(n_1, n_2) = 1$, *and let k_i be the order of* $2 \pmod{n_i}$, $i = 1,2$; *then* $C[n_1n_2] \geq C[n_1]C[n_2]$, *with equality if and only if* $\gcd(k_1, k_2) = 1$.

Spring Hops Eternal November 1980

1109. March 21 is commonly considered as the first day of spring (the date of the vernal equinox) —a tradition dating from the Council of Nicaea in 325 A.D. The most recent year in which this was in fact true was 1979, when the vernal equinox occurred at 12:22 a.m. EST on March 21.

When will be the next year in which spring begins as late as March 21 in the United States? (The average interval between vernal equinoxes—the tropical year—is to be taken as 365 days, 5 hours, 48 minutes, and 46 seconds.) [*Thomas R. Nicely, Lynchburg College.*]

Solution: Since the interval between vernal equinoxes is 365 days, 5 hours, 48 minutes, and 46 seconds, the equinox would occur 23 hours, 15 minutes, and 4 seconds later every four years if there were no leap years. However, due to leap years, the equinox will occur 44 minutes, 56 seconds earlier every four years until we reach the year 2100, which is not a leap year. Hence, not until 1:09:04 a.m. EST on March 21, 2103, will spring arrive on March 21st again in the United States.

PETER SCHUMER
University of Maryland

Also solved by Walter Bluger (Canada), Ragnar Dybvik (Norway), Milton Eisner, Daniel Finkel, Nick Franceschine III, Kent Harris, Harry Hickey, and the proposer.

Seven Is Possible **November 1980**

1110. A certain mathematician, in order to make ends meet, moonlights as an apprentice plumber. One night, as the mathematician contemplated a pile of straight pipes of equal lengths and right-angled elbows, the following question occurred to this mathematician: "For which positive integers n could I form a closed polygonal curve using n such straight pipes and n elbows?" [*Gerald Wildenberg, University of Hartford.*]

Solution: It can be done for any even number $\geqslant 4$, for any odd number $\geqslant 7$, and no others. If $n=2k+2, k\geqslant 1$, arrange k lengths in one plane (e.g., in a stairstep pattern), another k in a plane parallel to these, and join the two pairs of ends by pipes perpendicular to these planes. If $n=2k+5, k\geqslant 1$, arrange $2k+2$ of them as before, with one of the end joiners going from (0,0,0) to (1,0,0) (assuming unit length pipes), and the two adjoining pipes dropping vertically from the (x, y)-plane. Now remove this end joiner and close the polygon with four pipes described by the vectors $(a,b,0)$, $(-c,d,e)$, $(-c,-d,-e)$ and $(a,-b,0)$ placed end-to-end starting at (1,0,0) and ending at (0,0,0), where $a=(\sqrt{2}-1)/2$, $b=\sqrt{1+2\sqrt{2}}\,/2$, $c=\sqrt{2}/2$, $d=ac/b$, and $e=\sqrt{(9-4\sqrt{2})/7}$. That no five-sided solution exists may be seen as follows: without loss of generality we need vectors $v_1=(1,0,0)$, $v_2=(0,1,0)$, $v_3=(a,0,b)$, $v_4=(c,d,e)$ and $v_5=(0,f,g)$ with $\Sigma v_i=0$, $v_1\perp v_2\perp\cdots\perp v_5\perp v_1$ and each $|v_i|=1$. This gives the following eight conditions:

(1) $a+c=-1$	(2) $d+f=-1$
(3) $b+e+g=0$	(4) $ac+be=0$
(5) $df+eg=0$	(6) $a^2+b^2=1$
(7) $c^2+d^2+e^2=1$	(8) $f^2+g^2=1$.

Then $0=ac+be=a(-1-a)+b(-b-g)=-a-a^2-b^2-bg=-a-1-bg$, so $bg=-a-1\leqslant 0$ since $a^2\leqslant 1$. Since all components are $\leqslant 1$ in magnitude, (1) and (2) imply $a\leqslant 0$, $c\leqslant 0$, $d\leqslant 0$ and $f\leqslant 0$. Thus $ac\geqslant 0$ and by (4), $be\leqslant 0$. Also $df\geqslant 0$ so by (5) $eg\leqslant 0$, and therefore $bge^2\geqslant 0$. But $bg\leqslant 0$, so $bg=0$ or $e=0$. It is fairly easy to check that there are no solutions with b, e or g equal to 0. (In four dimensions a five-sided solution is not hard to find, however.)

G. A. HEUER
Concordia College

Editor's Comment. We received many claims that a solution was possible only for even $n\geqslant 4$. FIGURE 1 illustrates the crucial case of $n=7$. A model of this case is easily made from a chemistry molecule set or from children's construction toys. Also, the plastic pipe now in common usage by plumbers overcomes the objection of two readers with regard to threading problems with metal pipe.

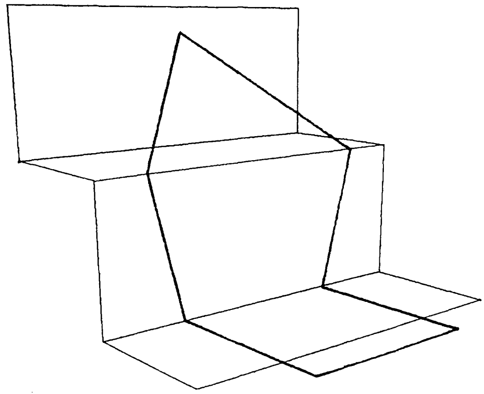


FIGURE 1

Also solved by Clayton Dodge, Henry J. Osner, J. L. Selfridge, University of Arizona Problem Solving Club, and the proposer.

$\int_0^x \sin(1/t)\,dt$ **January 1981**

1112. The function $\sin(1/t)$ is bounded and continuous everywhere except at 0, thus is (Riemann) integrable over any bounded interval. We may, therefore, define $F(x)$ to be $\int_0^x \sin(1/t)\,dt$, where x is any real number. Is F differentiable at 0? [*Richard Dowds, Fredonia State University College.*]

Solution: We shall show below that $|F(x)| \leqslant 3x^2$ for all x in $\mathbb{R}$. Consequently, we have that $\lim_{x\to 0} F(x)/x = 0$, and therefore $F'(0) = 0$ since $F(0) = 0$.

In order to establish the above inequality, we only need to consider $x > 0$, since $F(0) = 0$ and $F(x) = F(-x)$ for all x. Now, if $0 < y < x$, then

$$\begin{aligned}\int_y^x \sin(1/t)\,dt &= \int_y^x t^2 \frac{\sin(1/t)}{t^2}\,dt \\ &= x^2\cos(1/x) - y^2\cos(1/y) - \int_y^x 2t\cos(1/t)\,dt.\end{aligned}$$

Therefore,

$$\left|\int_y^x \sin(1/t)\,dt\right| \leqslant x^2 + y^2 + \int_0^x 2t\,dt \leqslant 3x^2.$$

Hence,

$$\begin{aligned}|F(x)| = \left|\int_0^x \sin(1/t)\,dt\right| &= \left|\lim_{y\to 0^+} \int_y^x \sin(1/t)\,dt\right| \\ &= \lim_{y\to 0^+} \left|\int_y^x \sin(1/t)\,dt\right| \leqslant 3x^2.\end{aligned}$$

STEVE RICCI
Boston College

Also solved by Nicolas Artemiadis (Greece), Richard Beigel, Larry F. Bennett & Kenneth L. Yocom, Artin Boghossian (Saudi Arabia), George Bridgman, Chico Problem Group, Edmund I. Deaton, Michael W. Ecker, M. B. Gregory, G. A. Heuer, Richard Johnsonbaugh, William J. Knight, L. Kuipers (Switzerland), Gloria Melara & Sergio Ruiz, Daniel A. Rawsthorne, C. Ray Rosentrater, J. M. Stark, Raimond A. Struble, John C. Tripp, Albert Wilansky, and the proposer.

It was pointed out by Wilansky and the proposer that this is essentially Problem E1071 *in the American Mathematical Monthly,* 1954, *p.* 124, *and by Jürg Rätz (Switzerland) that this is closely related to Monthly Problem* E1970, 1968, *p.* 678. *There were three incorrect solutions.*

1113. On Christmas Eve, 1983, Dean Jixon, the famous seer who had made startling predictions of the events of the preceding year, declared that the volcanic and seismic activities of 1980 and 1981 were connected with mathematics. The diminishing of this geological activity depended upon the existence of an elementary proof of the irreducibility of the polynomial $P(x)=x^{1981}+x^{1980}+12x^2+24x+1983$. Is there such a proof? [*William A. McWorter, Jr., The Ohio State University.*]

Solution I: To show that $P(x)$ is irreducible over the rationals, it suffices to do so over the integers, by Gauss's Lemma.

Suppose then that $P(x)=g(x)h(x)$, where g and h are nonconstant monic polynomials with integral coefficients. Modulo 3, $P(x)$ factors as $x^{1980}(x+1)$. Hence the reductions modulo 3 of $g(x)$ and $h(x)$ can be taken to have the forms x^{1980-k} and $x^{k+1}+x^k$, respectively, where $0\leqslant k<1980$. We then obtain $g(x)=x^{1980-k}+3A(x)$, where $\deg A<1980-k$, and $h(x)=x^{k+1}+x^k+3B(x)$, where $\deg B<k+1$. So

$$P(x)=x^{1981}+x^{1980}+3x^{1980-k}B(x)+3x^k(x+1)A(x)+9A(x)B(x).$$

If $k\neq 0$, then $1983=P(0)=9A(0)B(0)$, a contradiction, since $9\nmid 1983$. If $k=0$, then $\deg h=1$. However, $P(x)$ takes only odd values, and hence has no linear factors modulo 2, again a contradiction.

Clearly, this argument can be generalized to give the following variant of Eisenstein's Criterion: Let $P(x)=x^n+\alpha x^{n-1}+Q(x)$ be an integral polynomial, where $\deg Q\leqslant n-2$, and all coefficients of Q are divisible by a certain prime p, but $p^2\nmid Q(0)$. Suppose there is some prime that divides no value $P(n)$, n an integer. Then $P(x)$ is irreducible over the rationals.

WILLIAM H. GUSTAFSON
Texas Tech University

Solution II: We show that $P(x)$ is reducible over the field of integers modulo the prime 1979. Since

$$P(x)=x^{1981}+x^{1980}+12x^2+24x+1983=x^{1980}(x+1)+12x(x+2)+1983,$$

we have $P(-2)=-2^{1980}+1983$. Now, since 1979 is prime, we invoke Fermat's Theorem and obtain $2^{1979}\equiv 2 \pmod{1979}$. Hence $-2^{1980}\equiv -4 \pmod{1979}$ and so $P(-2)\equiv -4+1983\equiv 0 \pmod{1979}$; consequently $x+2$ divides $P(x) \pmod{1979}$.

KENNETH A. BROWN, JR.
Nova High School
Fort Lauderdale, Florida

Also solved (over the rationals) by Richard Beigel, Chico Problem Group, David Del Sesto, Robert Gilmer, E. F. Schmeichel, and the proposer; and (over another field) by Walter Bluger (Canada). There were two incorrect solutions.

Due to an editing error, the words "over the rationals" were omitted from the proposal statement. Del Sesto and Schmeichel found the appropriate generalization of the Eisenstein Criterion in Birkhoff and Mac Lane, "A Survey of Modern Algebra," fourth edition, p. 89, problem 4.

Answers

Solutions to the Quickies which appear near the beginning of the Problems section.

Q669. Although there is agreement when θ is a multiple of $\pi/2$, the θ in $x=4\cos\theta$, $y=3\sin\theta$ is not the polar angle. One way to see this is to observe that y/x is $(3\tan\theta)/4$ and not $\tan\theta$.

Q670. If $N<10^{100}$ and $10^{100}-7\leqslant 7N<10^{100}$, then $N=[10^{100}/7]=14285714285714\ldots$, that is, the digits of N are the same as the digits in the decimal expansion of $1/7$. The 50th digit is then 4 since $50\equiv 2 \pmod 6$.

REVIEWS

Paul J. Campbell, Editor
Beloit College

Pierre J. Malraison, Jr., Editor
MDSI, Ann Arbor

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of the mathematics literature. Readers are invited to suggest items for review to the editors.

Gardner, M., *Mathematical Games: Euclid's Parallel Postulate and its offspring*, Scientific American 245:4 (October 1981) 23-40.

A look at the history of Euclid's fifth postulate and the development of non-Euclidean geometry. Includes applications by M.C. Escher and Einstein.

Riegel, Peter S., *Athletic records and human endurance*, American Scientist 69 (May-June 1981) 285-290.

Fits simple power functions for time versus distance to world-record performances in various sports, justifies the fit over the range $3.5 \leq t \leq 230$ (min.), and draws conclusions from the model.

Tierney, John, *Sculpting by numbers*, Science 81 2:7 (September 1981) 78-79.

Describes how an artist and computer scientist spent six months revising the design of a sculpture, using color graphics.

Conner, T.M. and Hunt, B.R., *Image processing by computer*, Scientific American 245:4 (October 1981) 214-226.

Different techniques of computer enhancement of photographs. Applications are to reading blurred license plate numbers and analyzing Landsat data. Methods include changing color and reflective properties of pictures.

Sechret, S. and Greenberg, D., *A visible polygon reconstruction algorithm*, Computer Graphics 15:3 (August 1981) 17-27.

A new algorithm for the hidden surface problem, based on processing of the projected image on a raster screen.

Dill, J.C., *An application of color graphics to the display of surface curvature*, Computer Graphics 15:3 (August 1981) 153-161.

The application is intended to help car designers see where highlights will appear on a computer-designed body. It would also provide an excellent teaching tool and source of examples for advanced calculus.

Dunham, D., Lindgren, J. and Witte, D., *Creating repeating hyperbolic patterns*, Computer Graphics 15:3 (August 1981) 215-233.

A computer program for generating pictures à la "Circle Limit II" of M.C. Escher.

Kempe, Frederick, *Cube's Mr. Rubik is rich man but lives like poor professor; planning debut of new game, inventor lacks telephone, drives dilapidated car*, Wall Street Journal (23 September 1981) 1, 20.

Relates human details about Ernö Rubik and briefly describes his new inventions: Rubik's Snake, already available in the U.S., and Rubik's Game, "a social toy," to be out sometime in 1982. Despite large royalty earnings from the cube, Rubik resists taking on the trappings of riches, insisting: "The money is secondary. Only the creation counts."

Taylor, Don, Mastering Rubik's Cube: The Solution to the 20th Century's Most Amazing Puzzle, Holt, Rinehart & Winston, 1980; 31 pp, $1.95(P).

First steps in this succinct solution are given in prose, then the author switches to an algebraic notation.

Nourse, James G., The Simple Solution to Rubik's Cube, Bantam, 1981; 64 pp, $1.95(P).

Cube solution with a great many illustrations. No hint of any relation to mathematics.

Steen, Lynn Arthur, ed., Mathematics Tomorrow, Springer-Verlag, 1981; vi + 250 pp, $18.

Two dozen distinctive and thought-provoking essays about the nature of mathematics, the arts of teaching and learning it, issues of equality, and prospects for its future. A sequel to *Mathematics Today*, the intended audience this time is teachers of mathematics rather than the general public.

Aveni, Anthony F., Skywatchers of Ancient Mexico, U Texas Pr, 1980; xiii + 355 pp, $25.

Comprehensive introduction to Mesoamerican archaeoastronomy, replete with figures and diagrams.

Profiles in Applied Mathematics, SIAM, 1981; 32 pp, (single copy free), (P).

Booklet of 17 industrial profiles from *SIAM News*, plus a bibliography on careers in mathematics and related fields.

Smith, H.T. and Green, T.R.G., eds., Human Interaction with Computers, Acad Pr, 1980; x + 369 pp, (P).

How do we create responsive computers? What are the sociological implications of computer systems? What has been the impact of computers, and what is its potential, in learning, information retrieval, decision-making, design? How should programming languages be designed? If the computer is to serve humanity, and not vice versa, these are the important questions. The contributors to this book offer agreement, not on solutions, but at least on criteria to apply.

The Geosphere (TM), Finite Element Engineering (1629 Ohio Street, Flint, MI 48506); $40.

The Geosphere is a kit for making a truncated icosahedron a.k.a. a geodesic sphere. The kit consists of 180 plastic triangles, each of which belongs to one of two congruence classes. The sides of the triangles have alternate male and female connectors; within each congruence class there are two patterns of connectors along the sides. The completed sphere is a period three geodesic sphere of radius about one and a half feet. This kit comes with instructions for making a truncated icosahedron; other polyhedra are possible. Not as colorful as Rubik's cube, but a nice geometric puzzle to attempt assembling.

Haack, Dennis G., Statistical Literacy: A Guide to Interpretation, Duxbury, 1979; xii + 323 pp.

Deliberately sets out to emphasize the *thinking* involved in the interpretation of statistics, rather than calculation and data manipulation. Dispenses with formulas and symbolic language, concentrates on large-sample tests of proportion. An ideal text for elementary statistics, or as a companion to one more traditionally mathematical. Does not require any, nor does it enhance, mathematical background of the students; and hence a course based on it alone should not satisfy a mathematics requirement.

Boyce, William E., ed., Case Studies in Mathematical Modeling, Pitman, 1981; xi + 386 pp, $19.95.

Traffic flow, elevator systems, herbicide resistance, crystal growth, network paths, data communications and operating system security. Both deterministic and stochastic models, with emphasis on the model formulation rather than the mathematical solutions.

West, Bruce J., ed., Mathematical Models as a Tool for the Social Sciences, Gordon and Breach, 1980; 120 pp, $26.50.

Historiography and retrospective econometrics, serial memory, contract acceptance, extreme events, mating systems, political conditions, speculation and income distribution. As in most collections on mathematical modeling, facility in calculus and calculus-based probability and statistics is assumed.

Priorities in School Mathematics: Executive Summary of the PRISM Project, National Council of Teachers of Mathematics, 1981; 33 pp, single copy free (P).

Presents separate reactions to NCTM's *An Agenda for Action: Recommendations for School Mathematics of the 1980's*, from groups of teachers, supervisors, principals, school board presidents and PTA presidents. Some interesting results: PTA presidents strongly support requiring 3 or more years of mathematics for all students (34% are for 4 years, 33% for 3, 30% for 2, 3% for 1), but strongly oppose use of calculators for doing homework and strongly support increased time on drill and practice (both in opposition to preferences of teachers).

Willoughby, Stephen S., Teaching Mathematics: What Is Basic?, Occasional Paper 31, Council for Basic Education (725 15th St., NW, Washington, DC 20005), 1981; 45 pp, $2 (P).

"Thinking is the basic skill to be taught in school mathematics.... [Students] will need to master many subsidiary skills, but such mastery should never be mistaken for the basic goal." The author, director of mathematics education at NYU, provides an important and careful response to the "back to basics" movement, which is worth sharing with school board members, administrators, teachers, and parents.

Commission on the Education of Teachers of Mathematics, Guidelines for the Preparation of Teachers of Mathematics, National Council of Teachers of Mathematics, 1981; vii + 21 pp, $4 (P).

Sets out guidelines for knowledge and competence in mathematics, by teaching level; also has guidelines for humanistic and behavioral studies, teaching and learning theory, laboratory and clinical experience, practicum, competence and utilization of faculty, faculty involvement with the school, admission to teacher education programs, and other areas. Priced too high for a document that deserves wide distribution.

NEWS & LETTERS

COUNTEREXAMPLES TO INNER PRODUCT SPACE "IDENTITIES"

On reading the Note of N. Fowler III (this *Magazine*, March 1979, pp. 96-97) I recall another example which I use as a homework exercise:

Let V be the linear space of polynomials $P(X)$ with real coefficients, equipped with the usual inner product $(P|Q) = \int_0^1 P(X)Q(X)dX$. Let S be the subspace generated by 1, X^2, X^4,... and W be the subspace generated by X, X^3, X^5,... . We have $S \cap W = 0$. Counterexamples to the three identities

$$S \oplus S^\perp = V$$
$$(S^\perp)^\perp = S$$
$$(S_1 \cap S_2)^\perp = S_1^\perp + S_2^\perp$$

will be furnished once we show that $S^\perp = W^\perp = 0$, which follows from the Lemma below.

Lemma. If $\int_0^1 P(X)X^n dX = 0$ for all even (resp. odd) n, then $P(X) \equiv 0$.

Proof. Suppose $P(X) = a_0 + a_1X + \ldots + a_kX^k$. Writing out the integral in full, we see that it suffices to show the invertibility of the two $(k+1) \times (k+1)$ matrices

$$\begin{bmatrix} 1 & 1/2 & 1/3 & \ldots \\ 1/3 & 1/4 & 1/5 & \ldots \\ \vdots & \vdots & \vdots & \end{bmatrix} \; \& \; \begin{bmatrix} 1/2 & 1/3 & 1/4 & \ldots \\ 1/4 & 1/5 & 1/6 & \ldots \\ \vdots & \vdots & \vdots & \end{bmatrix}.$$

Both matrices are of the form

$$A = (1/(\alpha_i + \beta_j))$$

where $\alpha_i = 2(i-1)$, $\beta_j = j$ in the first matrix and $\alpha_i = 2i - 1$, $\beta_j = j$ in the second matrix. Induction yields det $A =$

$$\prod_{i=2}^{k+1} \prod_{i>j} (\alpha_i - \alpha_j)(\beta_i - \beta_j) / \prod_{i,j} (\alpha_i + \beta_j),$$

so det $A \neq 0$ in our special cases.

Man-Keung Siu
University of Hong Kong
Hong Kong, HONG KONG

CORRECTIONS TO 50-YEAR INDEX

In scanning "Mathematics Magazine: 50 Year Index, 1926-1977" I have noted these omissions:

Marlow Sholander, "Conic Reflections," 34(1960) 124a;

Marlow Sholander, "Del Lemma," 34 (1961) 373;

C.W. Trigg, "Victor Thebault, 1882-1960," 33(1960) 303.

These omissions are understandable since all of these items appeared on inside back covers.

Charles W. Trigg, "On a note by Lin," 40(1967) 28, is referred to only in items on pages 53 and 143, where it is attributed to Lin.

Charles W. Trigg
2404 Loring Street
San Diego, CA 92109

1000 = W. THAT A P. IS W.

For readers puzzled by this cryptic title to the solution of Proposal No. 1101 (this *Magazine*, November 1981, p. 271), the rough translation is "1000 equals the number of words that a picture is worth." Dan Eustice noticed the right side of the "equation" (the left side was to be filled in) as a puzzle in *Games* magazine (May-June 1981, pp. 25, 66).

L. F. Meyers
Problems Editor

ANOTHER LOOK AT PLAYING TIME

Readers of "A Model for Playing Time" (this *Magazine*, November 1981, pp. 247-250) will surely be interested in "Kinematics of tape-recording," *American Journal of Physics*, January 1981, pp. 81-83.

Murray Klamkin
University of Alberta
Edmonton, Alberta
CANADA T6G 2G1

PLAYING TIME: THE PAPER INDUSTRY

Green Bay is in northeast Wisconsin where paper manufacturing is a major industry. One of my students, an engineer in a firm which develops machinery for the paper industry, described the following problem to me.

As toilet paper is wound on a spindle, it must be drawn at constant speed through a perforation machine assuring equidistant perforations. With the speed of the paper and the desired tightness of wrap as parameters, the spindle must be designed to revolve appropriately. This problem is almost identical to the one discussed in "A Model for Playing Time" (this *Magazine*, November 1981, pp. 247-250) and can be solved using the methods introduced there.

Dan Kalman
University of Wisconsin
Green Bay, WI 54302

SHORT COURSE: TEACHING COMPUTER SCIENCE IN A MATH DEPARTMENT

The Ohio Section of the Mathematical Association of America announces a 1982 short course: *Teaching Computer Science in a Mathematics Department*, to be held June 8-11, 1982, at Denison University in Granville, Ohio. There will be discussions with invited individuals and panels on most or all of the following:

1) a review of ACM and CUPM recommendations;
2) training needed for faculty;
3) current computer science programs within mathematics departments;
4) the design of the first computer course;
5) hardware; and
6) course offerings needed for business, industry, and graduate school.

The registration fee is $30 and cost for board and room, including banquet and hospitality, is $65. For further information contact Professor Andrew Sterrett, Jr. or Professor Zaven Karian, Department of Mathematics, Denison University, Granville, Ohio 43023.

ANNUAL SUMMARY OF RESULTS: ANNUAL HIGH SCHOOL MATHEMATICS EXAMINATION

The MAA Committee on High School Contests wishes to inform college and university mathematicians that each year it publishes a summary of the results of its Annual High School Mathematics Examination, also known as the "MAA Contest." A number of colleges and universities already use this document, and others might wish to do so, to identify prospective students with outstanding mathematical talent. In 1981 (the Thirty-Second Annual Examination), 6,797 schools and 422,231 students registered for the exam.

For the purposes of this Examination, the U.S. and Canada are divided into a number of regions and subregions, and the Regional Coordinators often compile regional Honor Roll lists containing more students from their area than are listed in the National Summary. For instance, the Ohio Regional Coordinator puts out a very informative summary, listing student scores for various categories of schools, and many colleges in Ohio offer scholarships to Ohio students on the basis of these results. The lists also provide information to MAA Sections which recognize high-scoring students by the presentation of certificates and awards.

The Annual High School Mathematics Examination is given each year in early March; the 1982 examination date is Tuesday, March 9th. The National and Regional Summaries are generally available by June. To get the address of the Regional Coordinator for your area, and to order a copy of the National *Summary of Results and Awards* ($1.50 each), write to:

Prof. Walter E. Mientka
Executive Director
MAA Committee on High School Contests
University of Nebraska
917 Oldfather Hall
Lincoln, NE 68588

SOLUTIONS TO XXII INTERNATIONAL MATHEMATICAL OLYMPIAD

In September 1981 we published the six problems which proved to be easy fare for many of the participants in the IMO held in Washington, D.C., in July 1981. (For reports on the IMO, see Focus, *September-October 1981 and this* Magazine, *News and Letters, September 1981. A full report, with solutions and information on scores is available for 50¢ from Dr. Walter E. Mientka, 917 Oldfather Hall, Univ. of Nebraska, Lincoln, Nebraska 68588.)*

The solutions which follow have been prepared for publication in this Magazine *by Loren Larson, of St. Olaf College.*

1. P is a point inside a given triangle ABC. D, E, F are the feet of the perpendiculars from P to the lines BC, CA, AB, respectively. Find all P for which

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}$$

is least.

Sol. Denote the lengths BC, AC, AB, by a, b, c respectively, and, PD, PE, PF, by p, q, r respectively. We wish to minimize $a/p + b/q + c/r$. It is sufficient to minimize $(ap + bq + cr)(a/p + b/q + c/r)$ since $ap + bq + cr$ is twice the area of the triangle (a constant). The latter product equals $a^2 + b^2 + c^2 + ab(p/q + q/p) + bc(q/r + r/q) + ca(p/r + r/p)$. By the arithmetic mean-geometric mean inequality, $x + 1/x \geq 2\sqrt{x(1/x)} = 2$, for $x > 0$, with equality when $x = 1$. Thus, the preceding expression is minimized when $p = q = r$; that is, the minimum occurs when P is at the incenter and the minimum value is $(a + b + c)^2$.

2. Let $1 \leq r \leq n$ and consider all subsets of r elements of the set $\{1,2, \ldots,n\}$. Also consider the least number in each of these subsets. $F(n, r)$ denotes the arithmetic mean of these least numbers; prove that

$$F(n, r) = \frac{n+1}{r+1}.$$

Sol. There are $\binom{n-k}{r-1}$ subsets of r elements which have k as their least element, $k = 1,2,\ldots,n-r+1$. Therefore, $F(n,r) = N(n,r)/\binom{n}{r}$, where $N(n,r) = \binom{n-1}{r-1} + 2\binom{n-2}{r-1} + 3\binom{n-3}{r-1} + \ldots + (n-r+1)\binom{r-1}{r-1}$. One way to evaluate this sum is to write it in the form

$$\left[\binom{n-1}{r-1} + \binom{n-2}{r-1} + \ldots + \binom{r-1}{r-1}\right] + \left[\binom{n-2}{r-1} + \binom{n-3}{r-1} + \ldots + \binom{r-1}{r-1}\right] + \ldots + \left[\binom{r}{r-1} + \binom{r-1}{r-1}\right] + \left[\binom{r-1}{r-1}\right],$$

and to use the familiar summation formula,

$$\binom{p}{p} + \binom{p+1}{p} + \binom{p+2}{p} + \ldots + \binom{q}{p} = \binom{q+1}{p+1},$$

p and q integers, $p \leq q$, to get $N(n,r) = \binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \ldots + \binom{r}{r} = \binom{n+1}{r+1}$. It follows that $F(n,r) = \binom{n+1}{r+1}/\binom{n}{r} = \frac{n+1}{r+1}$.

3. Determine the maximum value of $m^2 + n^2$, where m and n are integers satisfying $m,n \in \{1,2,\ldots,1981\}$ and $(n^2 - mn - m^2)^2 = 1$.

Sol. Call an ordered pair (n,m) admissible if $n,m \in \{1,2,\ldots,1981\}$ and $(n^2 - nm - m^2)^2 = 1$. If $m = 1$, then $(1,1)$ and $(2,1)$ are the only admissible pairs.

For any admissible pair (n_1,n_2) with $n_2 > 1$, we have $n_1(n_1-n_2) = n_2^2 \pm 1 > 0$, so that $n_1 > n_2$. Define $n_3 = n_1 - n_2$. Then $1 = (n_1^2 - n_1n_2 - n_2^2)^2 = ((n_2+n_3)^2 - (n_2+n_3)n_2 - n_2^2)^2 = (-n_2^2 + n_2n_3 + n_3^2)^2$ so (n_2,n_3) is also an admissible pair.

If $n_3 > 1$, then, in the same way, we have $n_2 > n_3$, and, letting $n_2 - n_3 = n_4$ we have (n_3, n_4) is an admissible pair.

Thus we have a sequence (necessarily finite) $n_1 > n_2 > n_3 > \dots$ such that $n_{i+1} = n_{i-1} - n_i$ and where (n_i, n_{i+1}) is admissible for all i.

The sequence terminates if $n_i = 1$. Since (n_{i-1}, n_i) is admissible, and $n_{i-1} > 1$, $n_{i-1} = 2$ must hold. Therefore, (n_1, n_2) are consecutive terms of the truncated Fibonacci sequence 1597,987, ...,13,8,5,3,2,1. Conversely, any such pair is admissible, so the maximum value of $m^2 + n^2$ is $1597^2 + 987^2$.

4.(a) For which values of $n > 2$ is there a set of n consecutive positive integers such that the largest number in the set is a divisor of the least common multiple of the remaining $n - 1$ numbers?

(b) For which values of $n > 2$ is there exactly one set having the stated property?

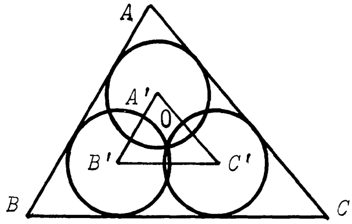
Sol. The answers to (a) and (b) are $n \geq 4$ and $n = 4$ respectively. To see this, let N denote the largest number in a set of n consecutive positive integers.

Suppose $n = 3$. Then $N \geq 3$, and it is clear that 2 is the only prime divisor of N. Thus $N = 2^a$ for some integer $a > 1$. But this is impossible since N-1 is odd and N does not divide N-2.

Suppose $n = 4$. Then N must have the form $2^a 3^b$ for some nonnegative integers a and b. It follows that 2^a must divide N-2, and this means that $a \leq 1$. Similarly, 3^b must divide N-3, and therefore $b \leq 1$. From this, it is easy to check that 6 is the only acceptable value for N.

Suppose $N > 4$. Then n-1 and n-2 are relatively prime, and both $N = (n-1)(n-2)$ and $N = (n-1)(n-2)/2$ are permissible values (e.g., N is larger than n, n-1 divides $N-(n-1)$ and n-2 divides $N-(n-2)$).

5. Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle, and the point O are collinear.



Sol. Let A', B', C' be centers of the three congruent circles with common point O which touch triangle ABC on sides AB and AC, BA and BC, CB and CA respectively. Corresponding sides of triangle $A'B'C'$ and triangle ABC are parallel. Furthermore, lines AA', BB', CC', bisect the angles at A,B,C (and A',B',C') respectively, and therefore they are concurrent at the incenter (of triangles ABC and $A'B'C'$). Therefore triangles ABC and $A'B'C'$ are homothetic with their center of similitude at the incenter.

Now, one of the fundamental properties of homothetic figures is that corresponding points lie on a line passing through the center of similitude. It follows that the circumcenter of triangle ABC, the circumcenter of triangle $A'B'C'$ (which is the point O), and the center of similitude (the incenter) are collinear.

6. The function $f(x, y)$ satisfies

(1) $f(0, y) = y+1$,

(2) $f(x+1, 0) = f(x, 1)$,

(3) $f(x+1, y+1) = f(x+1, y))$,

for all nonnegative integers x,y. Determine $f(4,1981)$.

Sol. One proves, by induction, that $f(1,y) = y + 2$, $f(2,y) = 2y + 3$, $f(3,y) = 2^{y+3}-3$, and $f(4,y) = \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{y+3\ 2\text{'s}} - 3$.

Hence, $f(4,1981) = \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{1984\ 2\text{'s}} - 3$.

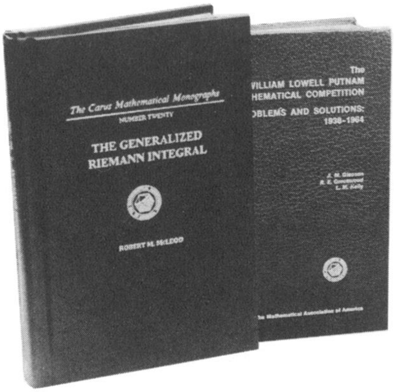
FOUR NEW BOOKS FROM THE MAA...

MAA Studies in Mathematics #20

STUDIES IN ALGEBRAIC GEOMETRY
A. Seidenberg, editor
143 + ix pages hardbound.
List: $16.00 MAA Member: $12.00

STUDIES IN ALGEBRAIC GEOMETRY, a new volume in the MAA Studies in Mathematics Series, contains expository articles which make some of the most interesting results in algebraic geometry accessible to students, as well as teachers and mathematicians who are nonspecialists or scientists working in other fields.

The contents include chapters on "initial results in the theory of linear algebraic groups," "the connectedness theorem and the concept of multiplicity," "space curves as ideal-theoretic complete intersections," and "Chasle's enumerative theory of conics."



Carus Mathematical Monographs #20

THE GENERALIZED RIEMANN INTEGRAL
by Robert M. McLeod
275 + xiii pages hardbound.
List: $18.00 MAA Member: $13.50

THE GENERALIZED RIEMANN INTEGRAL is the first book-length presentation of an exciting new development in integration theory. A new integral, a generalization of the familiar Riemann integral, has been discovered which has all the power and range of the Lebesque integral but which can be readily understood by anyone who has studied calculus through the sophomore level.

Because of the clarity and organization of the exposition and the inclusion of exercises designed to actively engage the reader in the material, it is eminently suitable for use as a textbook at the advanced undergraduate or beginning graduate level.

MAA Studies in Mathematics #21

STUDIES IN FUNCTIONAL ANALYSIS
Robert Bartle, editor
227 + xi pages hardbound.
List: $19.00 MAA Member: $14.00

STUDIES IN FUNCTIONAL ANALYSIS, a new addition to the MAA's series STUDIES IN MATHEMATICS, presents five expository articles on recent developments in functional analysis. Although these articles cannot provide a full account of the current state of the field, they do give a reader a tantilizing taste of some of the latest ideas and applications of this fascinating area of mathematics.

The contents include chapters on "numerical ranges," "projection operators in approximation theory," "complementably universal separable banach spaces; an application of counter examples to the approximation problem," "integral representation for elements of convex sets," and "aspects of banach lattices."

A New Problem Book from the MAA...

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION: PROBLEMS AND SOLUTIONS: 1938-1964
A. M. Gleason, R. E. Greenwood, and L. M. Kelly
652 + xi pages hardbound.
List: $35.00 MAA Member: $26.00

Problemists the world over, including all past and future Putnam Competitors, will revel in "mastering the difficulties" posed by this collection of problems from the first 25 William Lowell Putnam Competitions. Solutions to all 347 problems are given. In some cases multiple solutions are included, some which contestants could reasonably be expected to find under examination conditions and others which are more elegant or utilize more sophisticated techniques. Valuable references and comments on many of the problems are presented.

ORDER FROM:
The Mathematical Association of America
1529 Eighteenth Street, N.W.
Washington, D.C. 20036

SOME INFORMATIVE AND USEFUL BOOKS FROM THE MAA...

Browse through this list and see if your library is missing some of these important books published by the Association.

APPLICATIONS OF UNDERGRADUATE MATHEMATICS IN ENGINEERING—written and edited by Ben Noble. A collection of articles prepared by engineers for the Committee on the Undergraduate Program in Mathematics. 364 pp. Hardbound.
List: $19.00. MAA Member: $13.00.

ANNOTATED BIBLIOGRAPHY OF EXPOSITORY WRITING IN THE MATHEMATICAL SCIENCES, prepared by M. P. Gaffney and L. A. Steen. An invaluable reference source of expository articles in mathematics. 282 pp. Paperbound.
List: $12.00. MAA Member: $8.00.

AN ANNOTATED BIBLIOGRAPHY OF FILMS AND VIDEOTAPES FOR COLLEGE MATHEMATICS, by David I. Schneider. An up-to-date listing of films and videotapes available for classroom use. 107 pp. Paperbound.
List: $9.00. MAA Members: $6.00.

A BASIC LIBRARY LIST FOR TWO-YEAR COLLEGES, prepared by the Committee on Basic Library Lists. A recommended library nucleus for two-year colleges. 66. pp. Paperbound.
List: $8.00. MAA Member: $6.00.

A BASIC LIBRARY LIST FOR FOUR-YEAR COLLEGES, prepared by CUPM. Presents listings of books and journals that should be in every college library. 106 pp. Paperbound.
List: $9.00. MAA Member: $6.50

THE CHAUVENET PAPERS. *A Collection of Prize Winning Expository Papers in Mathematics,* edited by James C. Abbott. Two-volumes of the collected prize winning Chauvenet Papers. Vol. 1—312 pp. Hardbound. Vol. 2—282 pp. Hardbound.
List: $21.00 each. MAA Member $16.00 each.
[Two volume sets
List: $36.00. MAA Member: $27.00.]

CRITICAL VARIABLES IN MATHEMATICS EDUCATION: *Findings from a Survey of the Empirical Literature,* by E. G. Begle. A joint publication of the MAA and the National Council of Teachers of Mathematics. List: $8.00. MAA Member: $6.40.

A COMPENDIUM OF CUPM RECOMMENDATIONS. Volumes I and II. A collection of the major recommendations of the Committee on the Undergraduate Program in Mathematics. Two volumes. 756 pp. Hardbound.
List: $16.50. MAA Member: $12.00.

THE WILLIAM LOWELL, PUTNAM MATHEMATICAL COMPETITION: PROBLEMS AND SOLUTIONS—1938-1964. Compiled the R. E. Greenwood, A. M. Gleason, and L. M. Kelly. Contains problems and solutions to the first 25 Putnam Exams. 652 pp. Hardbound.
List: $35.00. MAA Member: $26.00.

THE MATHEMATICAL ASSOCIATION OF AMERICA: *Its First Fifty Years.* An historical perspective of the Association. 170 pp. Hardbound.
List: $10.00. MAA Member: $5.00.

FIFTY YEAR INDEX OF THE MATHEMATICS MAGAZINE, edited by Lynn A. Steen, and J. Arthur Seebach. Cumulative index of volumes 1-50. 163 pp. paperbound.
List: $10.00. MAA Member: $6.50.

INDEX OF THE AMERICAN MATHEMATICAL MONTHLY. Contains the tables of contents for each issue of volumes 1-80 as well as subject and author indices. 269 pp. Hardbound.
List: $19.00. MAA Member: $13.00.

PRIME-80. *Proceedings of a Conference on Prospects in Mathematics Education in the 1980's.* Included are the background of the conference and the recommendations that resulted.
84 pp. Paperbound. $3.50.

PROFESSIONAL OPPORTUNITIES IN THE MATHEMATICAL SCIENCES, Tenth Edition. 1978. Designed for the student interested in a career in mathematics. 35 pp. Paperbound. $1.50 each. 95¢ for orders of five or more.

RECOMMENDATIONS ON A GENERAL MATHEMATICAL SCIENCES PROGRAM. Prepared by the Committee on the Undergraduate Program in Mathematics. (CUPM) 102 pp. Paperbound. $3.50 each.

Order From:

THE MATHEMATICAL ASSOCIATION OF AMERICA
1529 Eighteenth Street, N.W.
Washington, D.C. 20036

Eminent Mathematicians and Mathematical Expositors speak to

STUDENTS and TEACHERS in . . .

The NEW MATHEMATICAL LIBRARY

An internationally acclaimed paperback series providing:

- stimulating excursions for students beyond traditional school mathematics.
- supplementary reading for school and college classrooms.
- valuable background reading for teachers.
- challenging problems for solvers of all ages from high school competitions in the US and abroad.

The New Mathematical Library is published by the MATHEMATICAL ASSOCIATION OF AMERICA. The volumes are paperbound.

For information regarding the price of these publications, please contact The Mathematical Association of America at the address listed below.

NUMBERS: RATIONAL AND IRRATIONAL by Ivan Niven, NML-01

WHAT IS CALCULUS ABOUT? By W. W. Sawyer, NML-02

AN INTRODUCTION TO INEQUALITIES, by E.F. Beckenbach, and R. Bellman, NML-03

GEOMETRIC INEQUALITIES, by N.D. Kazarinoff, NML-04

THE CONTEST PROBLEM BOOK. Problems from the Annual High School Mathematics Contests sponsored by the MAA, NCTM, Mu Alpha Theta, The Society of Actuaries, and the Casualty Actuarial Society. Covers the period 1950-1960. Compiled and with solutions by C.T. Salkind. NML-05

THE LORE OF LARGE NUMBERS, by P.J. Davis, NML-06

USES OF INFINITY, by Leo Zippin, NML-07

GEOMETRIC TRANSFORMATIONS, by I.M. Yaglom, translated by Allen Shields, NML-08

CONTINUED FRACTIONS, by C.D. Olds, NML-09

GRAPHS AND THEIR USES, by Oystein Ore, NML-10

HUNGARIAN PROBLEM BOOKS I and II, based on the Eötvos Competitions 1894-1905 and 1906-1928. Translated by E. Rapaport, NML-11 and NML-12

EPISODES FROM THE EARLY HISTORY OF MATHEMATICS, by A. Aaboe, NML-13

GROUPS AND THEIR GRAPHS, by I. Grossman and W. Magnus, NML-14

THE MATHEMATICS OF CHOICE, by Ivan Niven, NML-15

FROM PYTHAGORAS TO EINSTEIN, by K.O. Friedrichs, NML-16

THE CONTEST PROBLEM BOOK II. A continuation of NML-05 containing problems and solutions from the Annual High School Mathematics Contests for the period 1961-1965. NML-17

FIRST CONCEPTS OF TOPOLOGY, by W.G. Chinn and N.E. Steenrod, NML-18

GEOMETRY REVISITED, by H.S.M. Coxeter, and S.L. Greitzer, NML-19

INVITATION TO NUMBER THEORY, by Oystein Ore, NML-20

GEOMETRIC TRANSFORMATIONS II, by I.M. Yaglom, translated by Allen Shields, NML-21

ELEMENTARY CRYPTANALYSIS — A Mathematical Approach, by Abraham Sinkov, NML-22

INGENUITY BY MATHEMATICS, by Ross Honsberger, NML-23

GEOMETRIC TRANSFORMATIONS III, by I.M. Yaglom, translated by Abe Shenitzer, NML-24

THE CONTEST PROBLEM BOOK III. A continuation of NML-05 and NML-17, containing problems and solutions from the Annual High School Mathematics Contests for the period 1966-1972. NML-25

MATHEMATICAL METHODS IN SCIENCE, by George Pólya, NML-26

INTERNATIONAL MATHEMATICAL OLYMPIADS, 1959-1977. Problems, with solutions, from the first nineteen International Mathematical Olympiads. Compiled and with solutions by S.L. Greitzer. NML-27

THE MATHEMATICS OF GAMES AND GAMBLING, by Edward W. Packel, NML-28

THE MATHEMATICAL ASSOCIATION OF AMERICA

Send Orders to: **The Mathematical Association of America**
1529 Eighteenth St., N.W., Washington, D.C. 20036

HOUGHTON MIFFLIN UPDATE:

PCSP
Personalized Computational Skills Program, Volume 1
A Brush-Up Book™
Bryce R. Shaw, Senior author
Contributor: **Linda M. Oldaker**
Arizona State University
Advisers: **Ernest J. Cioffi**
University of Southern California
Sandra Clarkson, Hunter College, New York
Gwen Shelton, Bee College, Texas
Module A: Whole Numbers
192 perforated pages • paper
Module B: Fractions
192 perforated pages • paper
Module C: Decimals, Proportions, Percents
160 perforated pages • paper
Complete Three-Module Text
544 perforated pages • paper • Instructor's Manual including Alternate Testing Program
1980

PCSP
Personalized Computational Skills Program, Volume 2
A Brush-Up Book™
Bryce R. Shaw, Gwen Shelton, and **Sandra Clarkson**
Contributors: **Richard N. Aufmann** and **Vernon C. Barker,** both of Palomar College
Adviser: **Ernest J. Cioffi**
Module D: Applications Using Computational Skills: 192 perforated pages • paper
Module E: Special Skills with Applications
160 perforated pages • paper
Module F: Beginning Algebra with Applications: 160 perforated pages • paper
Complete Three-Module Text
512 perforated pages • paper • Instructor's Manual including Alternate Testing Program
1981

PCSP uses skill models, worked examples, and thousands of practice problems and applications in six step-by-step learning modules. Built-in diagnostic and mastery tests pinpoint students' problem areas and measure their learning progress. The two volumes may be used independently, concurrently, or in sequence.

Basic College Mathematics: An Applied Approach
Second Edition
Richard N. Aufmann and **Vernon C. Barker**
Both of Palomar College
About 512 pages • paper • Instructor's Manual • Solutions Manual • Early 1982

Organized into twelve units, Aufmann and Barker's new edition presents objectives for each section of each unit and criterion-referenced tests based on the objectives.

Interactive exercises for each objective, extensive drill, and numerous applied problems help develop skills. Other features include increased consumer applications and new material on using calculators in problem solving.

Basic Mathematics, Third Edition
M. Wiles Keller, Purdue University
James H. Zant, Oklahoma State University
634 perforated pages • paper
Instructor's Annotated Edition • 1979

Basic Mathematics: Skills and Structure
John F. Haldi, Spokane Community College
397 pages • paper • Instructor's Manual
1978

Form and Style: Theses, Reports, Term Papers
Sixth Edition
William Giles Campbell
Stephen Vaughan Ballou
Carole Slade, Columbia University
About 208 pages • paper • spiralbound
Just published

For adoption consideration, request examination copies from your regional Houghton Mifflin office.

Houghton Mifflin

Dallas, TX 75234 Geneva, IL 60134
Hopewell, NJ 08525 Palo Alto, CA 94304
Boston, MA 02107

THE MATHEMATICAL ASSOCIATION OF AMERICA
1529 Eighteenth Street, N.W.
Washington, DC 20036

MATHEMATICS MAGAZINE VOL. 55, NO. 1, JANUARY 1982